WaFu Notes
Discussions around the cold plasma model

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These notes correspond - more or less - to the presentation I gave at the WaFu summer school on July 6th, 2017 in Paris. They include several elements from discussions that took place at the summer school and helped clarify various points. The material is fairly standard, although I've chosen here the simplest possible model, hoping to emphasize the ideas rather than the results.

Topic: Dispersion relation for the cold plasma model, towards the notion of accessibility

The curious reader can refer to the following two books:

- PLASMA WAVES
  by D. G. Swanson
- RAY TRACING AND BEYOND : PHASE SPACE METHODS IN PLASMA WAVE THEORY
  by E. R. Tracy and A. J. Brizard

Disclaimer This is a first draft of these notes, there are most certainly mistakes in this document.

Feel free to report your comments to me !

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General plasma description

$$\begin{tikzcd}
\text{Electro-Magnetic fields} \arrow[r, bend left, "\text{Currents}"] & \text{Charged particles} \arrow[l, bend left, "\text{Lorentz force}"]
\end{tikzcd}$$

Particle models: hot, warm, cold plasma

1 Derivation of the cold plasma model

We derive here the single species cold plasma model for a homogeneous magnetized plasma, always considering that the frequency ω is a parameter.

1.1 Our hypotheses

- fluid model for the particles
- cold plasma: no thermal velocity, no pressure
- single species: electrons, velocity of ions is neglected
- uniform plasma: density n_0
- magnetized plasma: constant magnetic field $\mathbf{B}_0$
- different time scales: linearize around an equilibrium
- high frequency, Fourier in time: frequency ω

We think of $\mathbf{B}_0$ and n_0 as data, and of ω as a parameter

1.2 The PDE model

Notation: $\mathbf{x}$ space variable, t time, $e > 0$ charge of an electron, m_e mass of an electron, ϵ_0 vacuum permittivity, μ_0 vacuum permeability, $c^2 = 1/(\epsilon_0 \mu_0)$ speed of light in vacuum

$$\left\{ \begin{aligned} \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \partial_t \mathbf{E} \right) \\ m_e \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) &= -e \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \\ \mathbf{J} &= -e n_0 \mathbf{v} \end{aligned} \right. \tag{1}$$

Ansatz

$$\mathbf{Q}(\mathbf{x}, t) \leftarrow \mathbf{Q}_0(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) \exp(-\imath \omega t)$$

where $\mathbf{Q}_0$ is the equilibrium quantity, that is to say it is independent of t

Equilibrium $\mathbf{E}_0 = 0$, $\mathbf{v}_0 = 0$, $\mathbf{B}_0 \neq 0$ and constant since we consider a uniform plasma

Maxwell Eliminate $\mathbf{B}$

$$\nabla \times \nabla \times \mathbf{E} = \imath \omega \mu_0 \mathbf{J} + \frac{\omega^2}{c^2} \mathbf{E}$$

Euler Linearize

$$-\imath\omega m_e \mathbf{v} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B}_0$$

Write $\mathbf{B}_0 = B_0\mathbf{b}$

$$\left[-\imath\omega\mathbb{I} + \frac{eB_0}{m_e}\mathbf{b}\times\right]\mathbf{v} = \frac{-e}{m_e}\mathbf{E} \tag{2}$$

Definition 1. The cyclotron frequency is defined as

$$\omega_c = \frac{eB_0}{m_e}.$$

Remark 1. $\mathbf{v}$ and $\mathbf{E}$ are vector fields in the physical 3D space. The relation between $\mathbf{v}$ and $\mathbf{E}$ is linear, not only as a vector field but in a stronger sense as appears in (2): point-wise, that is to say that at any given point in space the relation between the velocity and electric field, as vectors in $\mathbb{C}^3$, is linear.

The data $\mathbf{B}_0$ introduces anisotropy in the model, therefore it is natural to choose an orthonormal basis of the physical space related to $\mathbf{B}_0$, rather than any orthonormal basis. Let's consider a basis whose z-axis is parallel to $\mathbf{b}$, so that (2) gives

$$\mathbf{v} = \frac{-\imath e}{m_e}\underbrace{\begin{pmatrix} \frac{\omega}{\omega^2-\omega_c^2} & \frac{-\imath\omega_c}{\omega^2-\omega_c^2} & 0 \\ \frac{\imath\omega_c}{\omega^2-\omega_c^2} & \frac{\omega}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & \frac{1}{\omega} \end{pmatrix}}_{:=\mathbb{S}}\mathbf{E} \tag{3}$$

<u>Back to Maxwell</u> (point-wise) linear constitutive relation $\mathbf{J}$ as a function of $\mathbf{E}$:

$$\mathbf{J} = \imath\frac{e^2 n_0}{m_e}\mathbb{S}\mathbf{E}$$

Eliminating the current field leads to the following second order PDE for the eletric field:

$$\nabla\times\nabla\times\mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} + \frac{\omega}{c^2}\frac{e^2 n_0}{m_e\epsilon_0}\mathbb{S}\mathbf{E} = 0$$

Definition 2. The plasma frequency $\omega_p > 0$ is defined as

$$\omega_p^2 = \frac{e^2 n_0}{m_e\epsilon_0}.$$

Definition 3. The dielectric tensor $\mathbb{K}$ is defined as

$$\mathbb{K} := \mathbb{I} - \frac{\omega_p^2}{\omega^2}\mathbb{S}.$$

Defining

$$S = 1 - \frac{\omega_p^2}{\omega}\left(\frac{\omega}{\omega^2-\omega_c^2}\right)$$

$$D = -\frac{\omega_p^2}{\omega}\left(\frac{\omega_c}{\omega^2-\omega_c^2}\right)$$

$$P = 1 - \frac{\omega_p^2}{\omega^2}$$

we have

$$\mathbb{K} = \begin{pmatrix} S & -\imath D & 0 \\ \imath D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

Remark 2. $\mathbb{K}$ is hermitian. The eigenvalues of $\mathbb{K}$ are P, $S+D$ and $S-D$.

Definition 4.

$$R := S + D$$

$$L := S - D$$

So

$$R = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\left(1 + \frac{\omega_c}{\omega}\right) = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)},$$

$$L = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\left(1 - \frac{\omega_c}{\omega}\right) = 1 - \frac{\omega_p^2}{\omega(\omega + \omega_c)},$$

and the sign of each eigenvalue clearly depends on the parameter ω.

<u>The model</u> for the unknown $\mathbf{E}$:

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2}\mathbb{K}\mathbf{E} = 0 \tag{4}$$

Remark 3. We will keep in mind that $\omega_p^2 \approx n_0$ while $\omega_c \approx B_0$, and that ω is a parameter: $\mathbb{K}$ depends on ω.

Remark 4. Because of the anisotropy in $\mathbb{K}$, the magnetic field $\mathbf{E}$ is expected to have different properties in different directions. In our model the anisotropy is due to the presence of an equilibrium magnetic field $\mathbf{B}_0$, so we will distinguish the parallel and perpendicular directions with respect to $\mathbf{B}_0$.

Remark 5. $\mathbb{S}$ and $\mathbb{K}$ are 3×3 matrices, $\mathbb{I}$ is the 3×3 identity matrix

Remark 6. Even though we did not use it here, we also have

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{5a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5b}$$

where ρ is the charge density

2 Dispersion relation

Looking for plane wave solutions to the PDE: $\mathbf{E} = \mathbf{A}\exp \imath\mathbf{k}\cdot\mathbf{x}$

- corresponds to Fourier transform in space
- such solutions, if they propagate, are called propagation modes
- we will give the definition of cut-off and resonance
- here we choose to look for $\mathbf{k}$ as a function of the parameter ω, while ω_c and ω_p are data

2.1 A matrix kernel problem

From the PDE

$$-\mathbf{k} \times \mathbf{k} \times \mathbf{A} - \frac{\omega^2}{c^2}\mathbb{K}\mathbf{A} = 0$$

Rescaling $\mathbf{n} = \frac{c}{\omega}\mathbf{k}$

$$[\mathbf{n} \times \mathbf{n} \times +\mathbb{K}]\mathbf{A} = 0 \tag{6}$$

Remark 7. So far the z-axis of our basis of the physical space has been fixed, parallel to $\mathbf{B}_0$, so the x- and y- axes could be anything. To simplify the expression of the matrix in square brackets, which depends on $\mathbf{n}$, we will now specify the components that are perpendicular to $\mathbf{B}_0$: we consider the x-axis so that $\mathbf{n}$ lies in the (x, z) plane.

Definition 5. Let θ be, in the (x, z) plane, the angle between $\mathbf{B}_0$ and $\mathbf{n}$, also called the incident angle of the wave. The parallel and perpendicular components, with respect to $\mathbf{B}_0$, of $\mathbf{n}$ are denoted

$$n_\parallel = n_z = n\cos\theta$$

$$n_\perp = n_x = n\sin\theta$$

Remark 8. It is **crucial** to see that $\mathbf{n}$(*or* $\mathbf{k}$) $\in \mathbb{R}^3$ corresponds to wave propagation, while $\mathbf{n}$(*or* $\mathbf{k}$) $\in \mathbb{C}\backslash\mathbb{R}^3$ corresponds to evanescent waves: indeed if $\mathbf{n}$ has a non-zero imaginary part, there is a real exponent in the exponential term of the propagation mode, which gives either an exponentially increasing solution, discarded for not being physical, or an exponentially decreasing solution, also called evanescent wave. Therefore, in what follows,

- $n^2 > 0$ corresponds to wave propagation,
- $n^2 < 0$ corresponds to evanescent waves.

Definition 6. Define the matrix $\mathcal{M}$ by

$$\mathcal{M}(n^2, \theta) = \begin{pmatrix} S - n^2\cos^2\theta & -\imath D & n^2\cos\theta\sin\theta \\ \imath D & S - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P - n^2\sin^2\theta \end{pmatrix}$$

Non trivial solutions of (6) exist if and only if $\det\mathcal{M}(n^2, \theta) = 0$, and the corresponding eigenvectors are the propagation modes.

Definition 7. If (n_*^2, θ_*) is such that $\det\mathcal{M}(n_*^2, \theta_*) = 0$ and if $\mathbf{A}_*$ is a null vector of $\mathcal{M}(n_*^2, \theta_*)$, then $\mathbf{E}_* = \mathbf{A}_* \exp \imath n_*(\sin\theta_* x + \cos\theta_* z)$ is called a propagation mode.

Remark 9. Since a propagation mode propagates if and only if $n_*^2 > 0$, see Remark 8, a cut-off frequency is a transition frequency between propagating frequencies and evanescent frequencies.

Remark 10. The definitions of n_*, θ_*, $\mathbf{A}_*$, and therefore of a propagation mode depend on the parameter ω - as well as the values of $\mathbf{B}_0$ and n_0.

Definition 8. For a given propagation mode, a cut-off frequency ω_* is defined implicitly by the condition $n_*(\omega_*) = 0$ while a resonance frequency ω_* is defined implicitly by the condition $n_*(\omega_*) = \infty$. Moreover, since $\omega_p^2 = \frac{e^2 n_0}{m_e \epsilon_0}$, the cut-off density corresponding to a frequency ω is defined by $n_{0,CO}(\omega) := \frac{m_e \epsilon_0 \omega}{e^2}$.

Remark 11. For any value of θ, $\mathcal{M}(0,\theta) = \mathbb{K}$. Therefore a cut-off frequency is equivalently defined implicitly as a frequency ω_* such that an eigenvalue of $\mathbb{K}(\omega_*)$ vanishes. It is independent of θ.

Remark 12. For various applications, different definitions of a cut-off can be used, such as the two following ones:

- for a given $(n_*)_\perp$, a cut-off frequency is defined implicitly by the condition $(n_*)_\parallel(\omega_*) = 0$,
- for a given $(n_*)_\parallel$, a cut-off frequency is defined implicitly by the condition $(n_*)_\perp(\omega_*) = 0$.

Plan For specific and general values of θ, we will now follow the following steps:

1. find the n^2 as a function of ω
2. find a corresponding polarization $\mathbf{A}$
3. identify cut-offs and resonances of the corresponding propagation mode
4. represent n^2 as a function of ω

We will use sub-indices for n referring to names of the different propagation mode, following the literature.

2.2 Parallel propagation

The case $\theta = 0$ (or equivalently $\theta = \pi$), which means that $\mathbf{n} \parallel \mathbf{B}_0$, is referred to as parallel propagation. Since

$$\mathcal{M}(n^2, \theta = 0) = \begin{pmatrix} S - n^2 & -\imath D & 0 \\ \imath D & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix}$$

we have $\det \mathcal{M}(n^2, \theta = 0) = P((S - n^2)^2 - D^2)$, so that

$$\det \mathcal{M}(n^2, \theta = 0) = 0 \Leftrightarrow n^2 = S \pm D = R \text{ or } L. \tag{7}$$

There are therefore two possibilities for n^2.

2.2.1 The right-handed wave

The first possibility corresponds to the plus sign in (7).

1. $n_R^2 := R = S + D$

$$\Rightarrow n_R^2 = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)}$$

2. $\mathcal{M}(n_R^2, \theta = 0) = \begin{pmatrix} -D & -\imath D & 0 \\ \imath D & -D & 0 \\ 0 & 0 & P \end{pmatrix} \Rightarrow \mathcal{M}(n_R^2, \theta = 0) \cdot \begin{pmatrix} -\imath \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \mathbf{A}_R := \begin{pmatrix} -\imath \\ 1 \\ 0 \end{pmatrix}$$

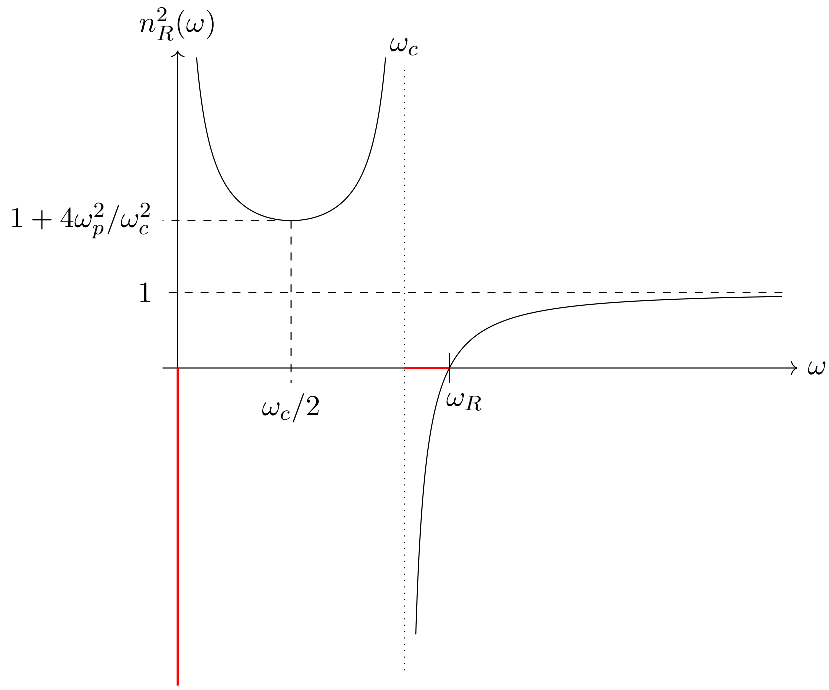

Figure 1: R-mode dispersion relation

3. <u>Resonances</u> There is a non-trivial resonance for $\omega = \omega_c$, called the electron cyclotron resonance, and a trivial one for $\omega = 0$

   <u>Cut-off</u> $n_R^2 = 0 \Leftrightarrow \omega^2 - \omega_c\omega - \omega_p^2 = 0 \Leftrightarrow \omega = \frac{\omega_c}{2} \pm \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2}$ so there is a single (> 0) cut-off frequency defined by

$$\omega_R := \frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2}$$

   Note that $\omega_R > \omega_c$ as long as $\omega_p^2 > 0$.

4. $n_R^2(\omega)$ has a local minimum for $\omega = \omega_c$, three asymptots, and is negative if and only if ω is between the resonance frequency ω_c and the cut-off frequency ω_R. See Figure 1.

So with $n_R := \sqrt{1 - \frac{\omega_p^2}{\omega(\omega-\omega_c)}}$, the vector field $\mathbf{E}_R := \mathbf{A}_R \exp \imath \frac{\omega}{c} n_R z$ is a solution to the PDE (4), referred to as the right-handed (or R) propagation mode. The R-mode does not propagate for $\omega \in [\omega_c, \omega_R]$. Since $\mathbf{A}_R \perp \mathbf{B}_0$, the R-mode solution has perpendicular polarization.

2.2.2 The left-handed wave

The second possibility corresponds to the minus sign in (7).

1. $n_L^2 := L = S - D$

$$\Rightarrow n_L^2 = 1 - \frac{\omega_p^2}{\omega(\omega + \omega_c)}$$

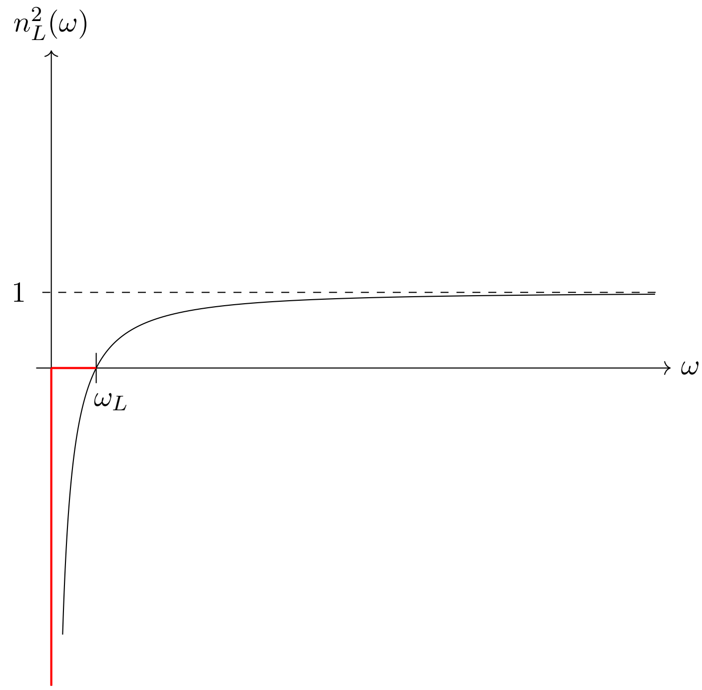

Figure 2: L-mode dispersion relation

2. $\mathcal{M}(n_L^2, \theta = 0) = \begin{pmatrix} D & -\imath D & 0 \\ \imath D & D & 0 \\ 0 & 0 & P \end{pmatrix} \Rightarrow \mathcal{M}(n_L^2, \theta = 0) \cdot \begin{pmatrix} \imath \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \mathbf{A}_L := \begin{pmatrix} \imath \\ 1 \\ 0 \end{pmatrix}$$

3. Resonances There is only a trivial one for $\omega = 0$

   Cut-off $n_L^2 = 0 \Leftrightarrow \omega^2 + \omega_c \omega - \omega_p^2 = 0 \Leftrightarrow \omega = -\frac{\omega_c}{2} \pm \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2}$ so there is a single (> 0) cut-off frequency defined by

$$\omega_L := -\frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2}$$

4. $n_L^2(\omega)$ is increasing and is negative if and only if ω is below the cut-off frequency ω_L. See Figure 2.

So with $n_L := \sqrt{1 - \frac{\omega_p^2}{\omega(\omega + \omega_c)}}$, the vector field $\mathbf{E}_L := \mathbf{A}_L \exp \imath \frac{\omega}{c} n_L z$ is a solution to the PDE (4), referred to as the left-handed (or L) propagation mode. The L-mode does not propagate for $\omega \leq \omega_L$. Since $\mathbf{A}_L \perp \mathbf{B}_0$, the L-mode solution has perpendicular polarization.

2.3 Perpendicular propagation

The case $\theta = \pi/2$ (or equivalently $\theta = -\pi/2$), which means that $\mathbf{n} \perp \mathbf{B}_0$, is referred to as perpendicular propagation. Since

$$\mathcal{M}(n^2, \theta = \pi/2) = \begin{pmatrix} S & -\imath D & 0 \\ \imath D & S - n^2 & 0 \\ 0 & 0 & P - n^2 \end{pmatrix}$$

we have $\det \mathcal{M}(n^2, \theta = \pi/2) = (P - n^2)((S - n^2)S - D^2)$, so that

$$\det \mathcal{M}(n^2, \theta = \pi/2) = 0 \Leftrightarrow n^2 = P \text{ or } (S^2 - D^2)/S. \tag{8}$$

There are therefore two possibilities for n^2.

2.3.1 The ordinary wave

The first possibility corresponds to the first option in (8).

1. $n_O^2 := P$

$$\Rightarrow n_O^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

2. $\mathcal{M}(n_O^2, \theta = \pi/2) = \begin{pmatrix} S & -\imath D & 0 \\ \imath D & S - P & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \mathbf{A}_O = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3. <u>Resonance</u> There is only a trivial one for $\omega = 0$

   <u>Cut-off</u> $n_O^2 = 0 \Leftrightarrow \omega^2 = \omega_P^2$

4. $n_O^2(\omega)$ is increasing and is negative if and only if ω is below the cut-off frequency ω_p. See Figure 3.

So with $n_O := \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$, the vector field $\mathbf{E}_O := \mathbf{A}_O \exp \imath \frac{\omega}{c} n_O x$ is a solution to the PDE (4), referred to as the ordinary (or O) propagation mode. The O-mode does not propagate for $\omega \leq \omega_p$. Since $\mathbf{A}_O \parallel \mathbf{B}_0$, the O-mode solution has parallel polarization.

2.3.2 The extra-ordinary wave

The second possibility corresponds to the second option in (8).

1. $n_X^2 := \frac{S^2 - D^2}{S} = \frac{RL}{S}$

$$\Rightarrow n_X^2 = \frac{[\omega(\omega - \omega_c) - \omega_p^2][\omega(\omega + \omega_c) - \omega_p^2]}{\omega^2(\omega^2 - \omega_c^2 - \omega_p^2)}$$

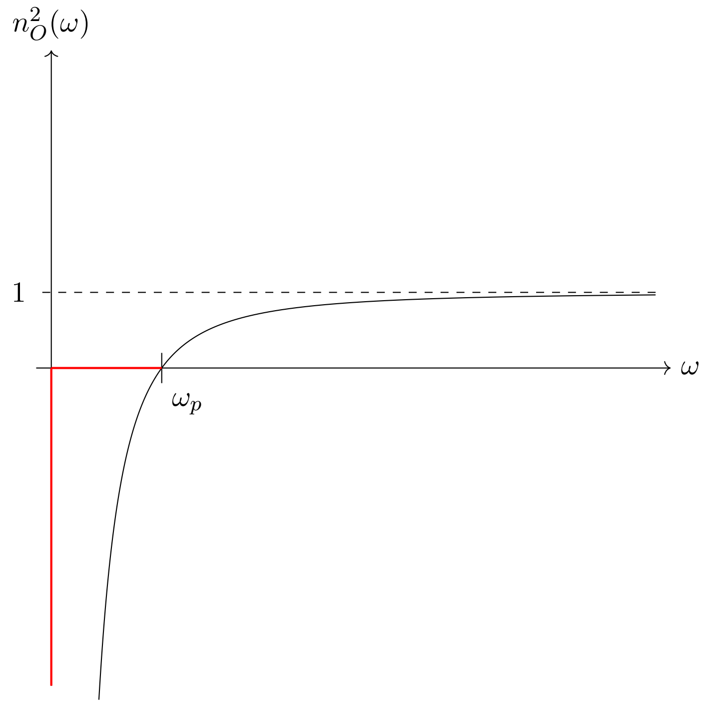

Figure 3: O-mode dispersion relation

2. $\mathcal{M}(n_X^2, \theta = \pi/2) = \begin{pmatrix} S & -\imath D & 0 \\ \imath D & \frac{D^2}{S} & 0 \\ 0 & 0 & P - \frac{RL}{S} \end{pmatrix} \Rightarrow \mathcal{M}(n_L^2, \theta = 0) \cdot \begin{pmatrix} 1 \\ -\imath \frac{S}{D} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \mathbf{A}_X = \begin{pmatrix} D \\ -\imath S \\ 0 \end{pmatrix}$$

3. Resonance There is a resonance defined by $S = 0 \Leftrightarrow \omega^2 - \omega_c^2 - \omega_p^2 = 0$ so there is a resonance frequency $\omega_{UH} > 0$ defined by

$$\omega_{UH}^2 = \omega_c^2 + \omega_p^2$$

and a trivial one for $\omega = 0$.

Cut-offs $n_X^2 = 0 \Leftrightarrow RL = 0$ so from 3 in 2.2.1 and 3 in 2.2.2 there are two cut-off frequencies, ω_R and ω_L.

4. Since

$$\frac{\omega_R^2 - \omega_{UH}^2}{2} = \frac{\omega_c}{2} \left(\sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2} - \frac{\omega_c}{2} \right) > 0$$

and

$$\frac{\omega_{UH}^2 - \omega_L^2}{2} = \frac{\omega_c}{2} \left(\frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2} \right) > 0,$$

it yields $\omega_L < \omega_{UH} < \omega_R$. Since we can write

$$n_X^2(\omega) = \frac{\omega^2(1 - \omega_p^2/\omega^2) - \omega_c^2}{\omega^2 - \omega_c^2 - \omega_p^2},$$

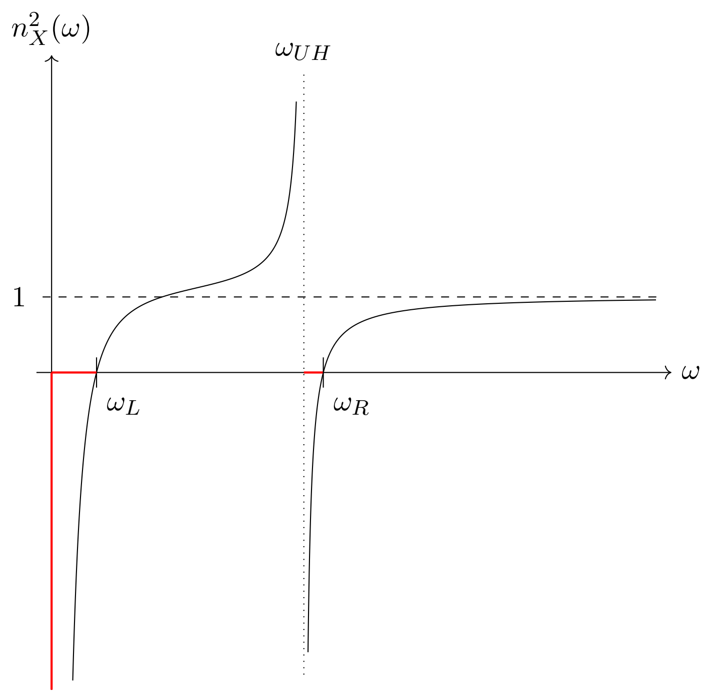

Figure 4: X-mode dispersion relation

let's denote by $\mathcal{D}$ the denominator, one can verify that

$$(n_X^2)'(\omega) \cdot \mathcal{D}^2 = 2\frac{\omega_p^2}{\omega^2} \cdot \frac{\omega^4 - 2\omega_p^2\omega^2 + \omega_p^2(\omega_p^2 + \omega_c^2)}{\omega} > 0,$$

which implies that $n_X^2(\omega)$ is increasing. this function is negative if and only if ω is below ω_L or between ω_{UH} and ω_R. See Figure 4.

So with $n_X := \sqrt{\frac{[\omega(\omega-\omega_c)-\omega_p^2][\omega(\omega+\omega_c)-\omega_p^2]}{\omega^2(\omega^2-\omega_c^2-\omega_p^2)}}$, the vector field $\mathbf{E}_X := \mathbf{A}_X \exp \imath\frac{\omega}{c} n_X x$ is a solution to the PDE (4), referred to as the extra-ordinary (or X) propagation mode. The X-mode does not propagate for $\omega \leq \omega_L$ as well as for $\omega \in [\omega_{UH}, \omega_R]$. Since $\mathbf{A}_X \perp \mathbf{B}_0$, the O-mode solution has perpendicular polarization.

2.4 Remarks on the case of general incidence angle

I the case of a general incidence angle θ, there are no such simple computation, but the following notation is used:

$$A = P\cos^2\theta + S\sin^2\theta$$

$$B = RL\sin^2\theta + PS(1 + \cos^2\theta)$$

$$C = PRL$$

so that $\det \mathcal{M}(n^2, \theta) = An^4 - Bn^2 + C$. The dispersion relation $\det \mathcal{M}(n^2, \theta) = 0$ is also written

$$\tan^2\theta = \frac{P(n^2 - R)(n^2 - L)}{(Sn^2 - RL)(n^2 - P)},$$

formula from which it is easy to retrieve the four cases studied in 2.2.1, 2.2.2, 2.3.1 and 2.3.2.

Cut-offs The general condition for a cut-off is $PRL = 0$, so in the general case the cut-off frequencies are ω_p, ω_L and ω_R.

3 Weakly inhomogeneous plasma

We consider now smooth and slow variations of the data $\mathbf{B}_0(\mathbf{x})$, $n_0(\mathbf{x})$, and in turn the frequencies ω_c, ω_p, ω_R, ω_L, ω_{UH} all depend on $\mathbf{x}$. Contrarily to the point of view chosen in the previous section, here the frequency ω is considered as data, while the parameter is the space variable $\mathbf{x}$. In certain cases the frequency ω is related to the antenna sending the wave from the chamber wall towards the plasma.

As $\mathbf{B}_0$ and n_0 depend on the space variable, the results obtained from taking the Fourier transform in space are not valid here. However, under the assumption of weak inhomogeneity, these results are used as a low order approximation.

Remark 13. Our goal here is not to give a mathematical justification for using results from the dispersion relation with variable coefficients, but rather to present some of the ideas appearing in the literature. So we will here apply the results of the previous section to the case of variable coefficients. The idea is rather to illustrate the **crucial** impact of the frequency of a wave, for a given propagation mode, on the sets in physical space where it can or cannot propagate. This is very related to the notion of accessibility, and mode conversion.

Remark 14. It won't be emphasized here, but of course the equilibrium data,$\mathbf{B}_0$ and n_0, also play an important role on wave propagation.

We will focus here on the representation of perpendicular propagation proposed in the presentation of Emanuele Poli: the question here is, for a given geometry, that is to say for given profiles of $\mathbf{B}_0(\mathbf{x})$ and $n_0(\mathbf{x})$, identify subdomains of the plasma where waves can propagate. This is referred to as the question of accessibility in the literature. Having in mind a simplified geometry for poloidal plane in a tokamak, we will consider in a 2D space a circular plasma in the (x, y)-plane and a magnetic field $\mathbf{B}_0$ in the z-direction, with $n_0(x, y) = n_{\max} - ((x - x_0)^2 + y^2)$ and of $B_0(x, y) = 1/x$, as represented in Figure 5.

In this 2D setting, cut-offs and resonances are now curves defined implicitly as level sets of a given function: $\{(x, y) \in \mathbb{R}^2 / \omega_*(x, y) = \omega\}$. Note that in a 3D setting they would be surfaces.

3.1 The O-mode wave

The 0-mode cut-off is defined by $\{(x, y) \in \mathbb{R}^2 / \omega_p(x, y) = \omega\}$, which depends on n_0 and not on B_0. Let $(\omega_p)_{\max}$ denote the maximum value of $\omega_p(\mathbf{x})$ in the plasma. There are two possible situations:

- if the parameter ω is above $(\omega_p)_{\max}$, then $n_O^2(x, y) > 0$ in the whole domain so the O-mode wave can propagate in the whole domain,
- if the parameter ω is below $(\omega_p)_{\max}$, then the cut-off curve is the curve implicitly defined by $\{(x, y) \in \mathbb{R}^2 / \omega_p(x, y) = \omega\}$, and the O-mode wave can propagate on the side of the curve where $\omega > \omega_p(x, y)$, where $n_O^2(x, y) > 0$, and can not propagate on the side of the curve where $\omega < \omega_p(x, y)$.

Here since $\omega_p(x, y)$ depends only on the distance between the center of the circular plasma and the point (x, y), the cut-off curve, when it exists, is a circle independently of the value of ω. However

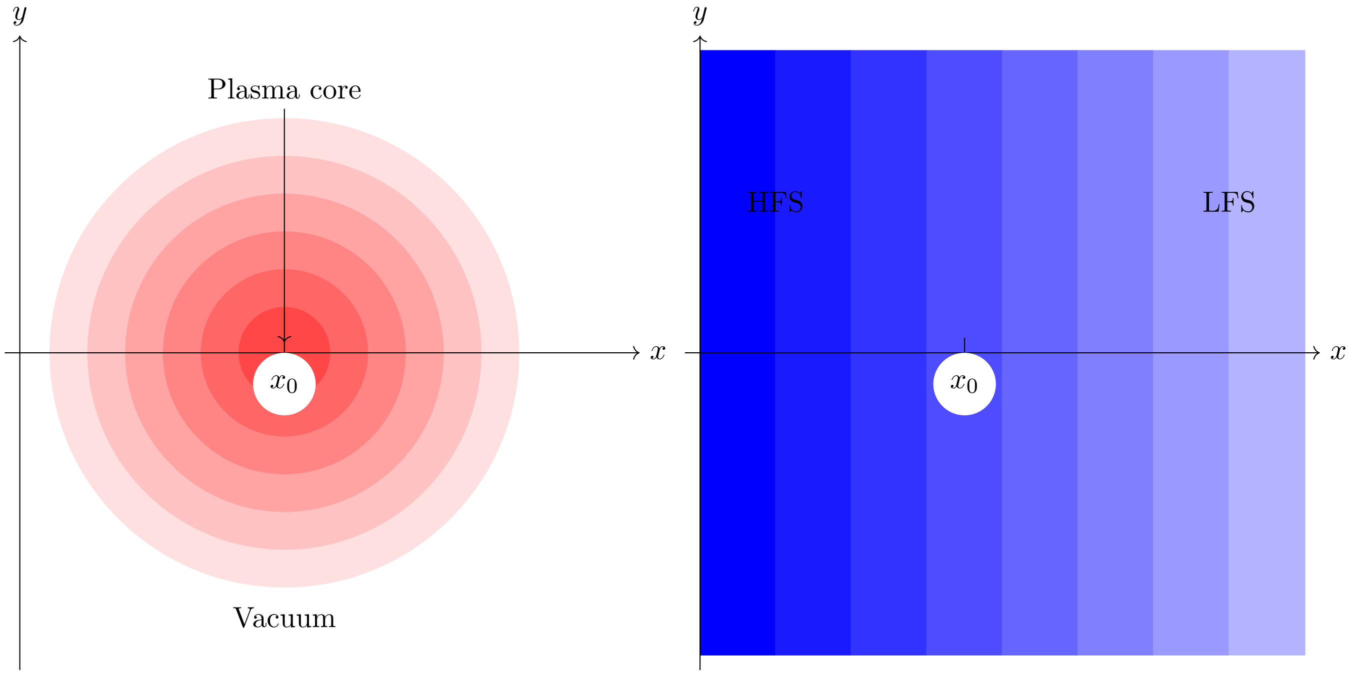

Figure 5: Level sets of $n_0(x,y) = n_{\max} - ((x-x_0)^2 + y^2)$ and of $B_0(x,y) = 1/x$, indicating the vacuum outside the plasma, the plasma core, the high field side (HFS) and low field side (LFS).

its radius depends on the value ω. Indeed if $C := e^2/(m_e\epsilon_0)$ then

$$\omega_p(x,y) = \omega \Leftrightarrow (x-x_0)^2 + y^2 = n_{\max} - \frac{\omega}{C}$$

See illustration on Figure 6.

3.2 The X-mode wave

For the X-mode wave, the resonance curve is defined by defined by $\{(x,y) \in \mathbb{R}^2/\omega_{UH}(x,y) = \omega\}$ and the cut-off curves are defined by $\{(x,y) \in \mathbb{R}^2/\omega_R(x,y) = \omega\}$ and $\{(x,y) \in \mathbb{R}^2/\omega_L(x,y) = \omega\}$. Let $(\omega_R)_{\max}$ and $(\omega_L)_{\max}$ denote the maximum values of $\omega_R(\mathbf{x})$ and $\omega_L(\mathbf{x})$ in the plasma. Here there are several possible situations, depending on the values of the parameter ω, but the situation is much more involved than for the O-mode wave. Here are some comments:

- if the parameter ω is above $(\omega_R)_{\max}$, then there are no cut-offs or resonances in the plasma, $n_X^2(x,y) > 0$ in the whole domain so the X-mode wave can propagate in the whole domain,
- if the parameter ω is above $(\omega_L)_{\max}$, then there is no L cut-off in the plasma,
- if the parameter ω is below $(\omega_L)_{\max}$, then the X-mode wave can not propagate on the side of the curve where $\omega < \omega_L(x,y)$, where $n_X^2(x,y) < 0$,
- at any point (x,y) such that the parameter ω is between $\omega_{UH}(x,y)$ and $\omega_R(x,y)$, the wave cannot propagate.

The situation the most commonly represented in the literature, including the two cut-offs and the resonance, is displayed here on Figure 7.

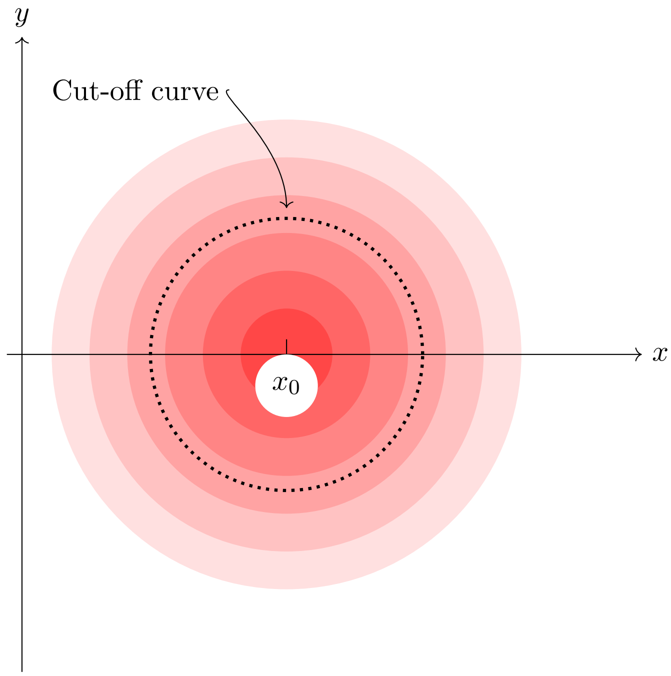

Figure 6: Example of a non-empty cut-off curve for the O-mode wave, that is to say for a frequency ω such that $\omega < \omega_p(x_0, 0) = (\omega_p)_{\max}$.

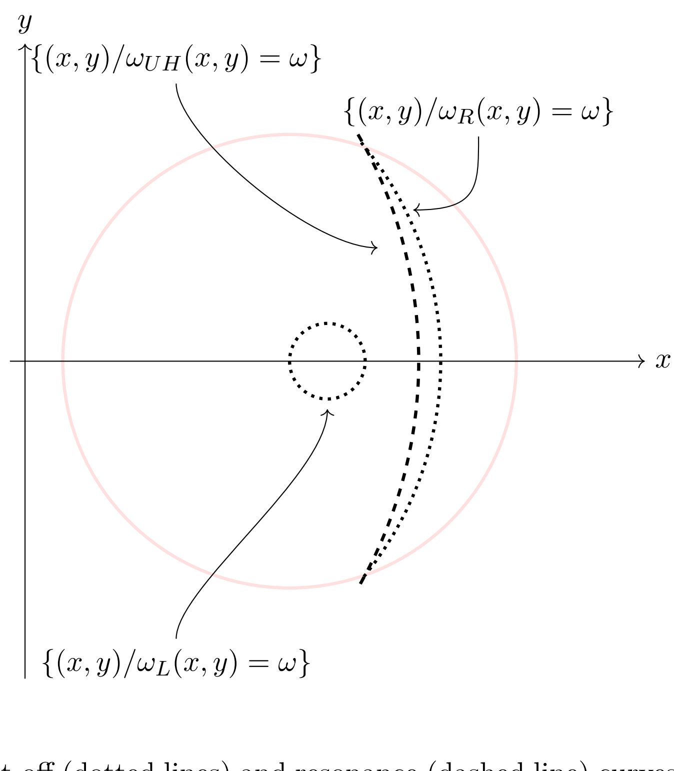

Figure 7: Example of cut-off (dotted lines) and resonance (dashed line) curves for the X-mode. The boundary of the plasma is represented in light red. The X-mode wave

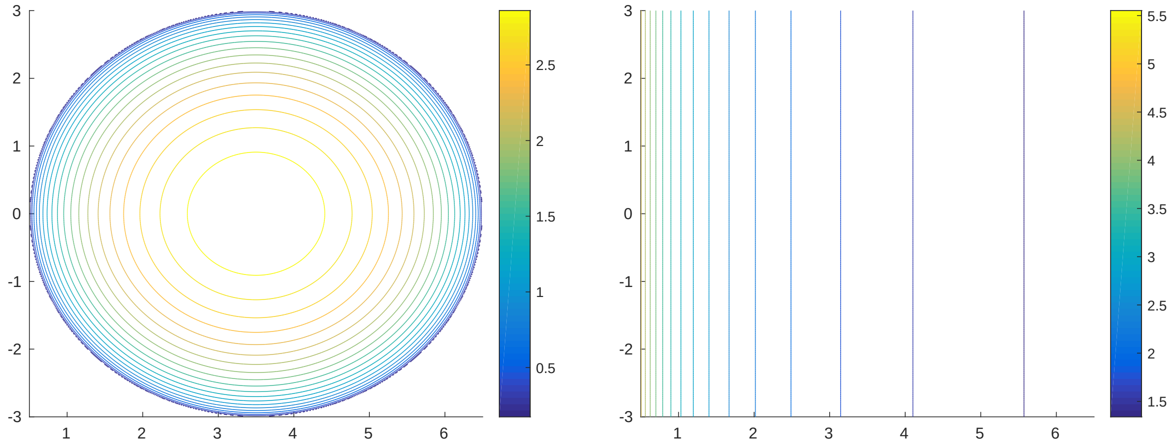

Figure 8: Level sets of the profiles of the plasma and cyclotron frequencies. Left: $\omega_p(x, y)$ defined in Equation (9). Right: $\omega_c(x, y)$ defined in Equation (10).

In order to illustrate the variety of possible situations for the X-mode wave, let's consider the profiles of $\omega_p(\mathbf{x})$ and $\omega_c(\mathbf{x})$:

$$\omega_p(\mathbf{x}) = \sqrt{9 - (x - 3.5)^2 - y^2}. \tag{9}$$

$$\omega_c(\mathbf{x}) = \frac{1}{0.01\sqrt{x}}, \tag{10}$$

There level sets are represented on Figure 8. Figure 9 displays the level sets of the corresponding cut-off and resonance frequencies $\omega_R(x, y)$, $\omega_L(x, y)$, $\omega_{UH}(x, y)$, using the same scale and highlighting simultaneously for these three functions the sets of level 1, 2.25, 3.45, and 4. The level sets of $\omega_L(x, y)$ are closed or empty, while the level sets of and $\omega_R(x, y)$ and $\omega_{UH}(x, y)$ can be empty but are not necessarily closed, or connected.

Finally, Figure 10 gathers, on single graphs representing the plasma, the non-empty level sets of a same level of the cut-off and resonance frequencies. On these graphs the white areas represent the zones where the X-wave can propagate, while the shaded areas represent the zones where the X-wave cannot propagate, either because $\omega < \omega_L(x, y)$ or because $\omega_{UH}(x, y) < \omega < \omega_L(x, y)$. These four values of ω chosen

$\omega = 1$ with only an L cut-off in the plasma, the X-wave can propagate only outside of the L cut-off curve, it cannot propagate inside of this curve curve

$\omega = 2.25$ with the two cut-offs and the resonance, the X-wave cannot propagate inside of the closed L cut-off curve, or between the UH resonance curve and the R cut-off curve, but can propagate otherwise

$\omega = 3.45$ with a two-component UH resonance and a single component R cut-off, the X-wave cannot propagate between the UH resonance curve and the R cut-off curve, but it can propagate inside the closed part of the resonance curve and outside the two open curves

$\omega = 4$ with a single component UH resonance and a two-component R cut-off, the X-wave cannot propagate inside the closed part of the R cur-off curve or between the cut-off and the resonance curves, but it can propagate between the two parts of the R cut-off curve and to the left of the resonance curve

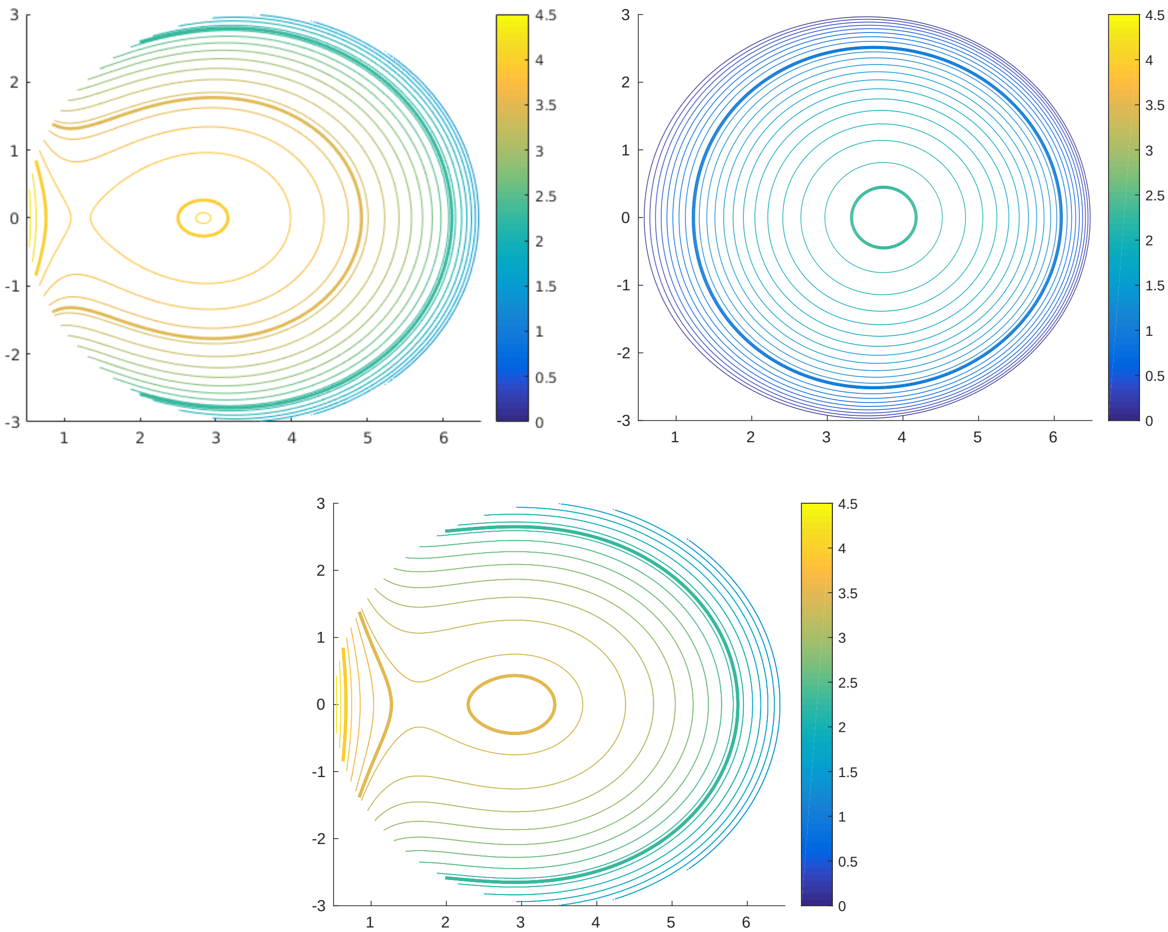

Figure 9: Level sets of the cut-off and resonance frequencies, computed for the ω_p and ω_c profiles given in (9) and (10). On the three parts of this figure, the same color scale is used, and four particular level sets are represented with a thick line, corresponding respectively to the levels 1, 2.25, 3.45, and 4, to emphasize the variety in topology of different level sets. Top left: $\omega_R(x,y)$. Top right: $\omega_L(x,y)$. Bottom: $\omega_{UH}(x,y)$.

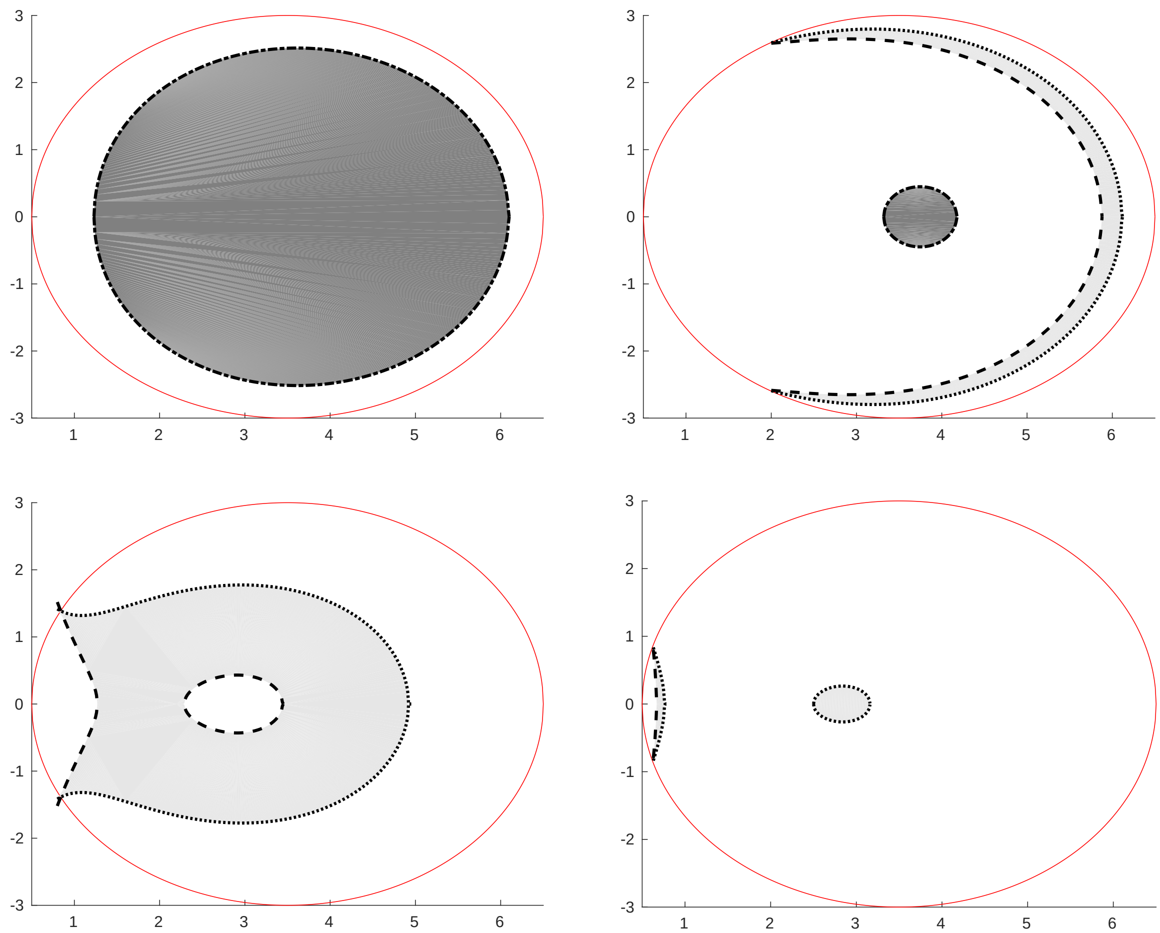

Figure 10: For four given values of ω, the sets $\{(x,y)/\omega_R(x,y)=\omega\}$, $\{(x,y)/\omega_L(x,y)=\omega\}$ and $\{(x,y)/\omega_{UH}(x,y)=\omega\}$ are represented, when not empty, on a same graph respectively in dotted line, dash-dot line and dashed line. They were computed for the ω_p and ω_c profiles given in (9) and (10). The shaded areas represent the zones where the X-wave cannot propagate: in dark for $\{(x,y)/\omega<\omega_L(x,y)\}$, in light for $\{(x,y)/\omega_{UH}(x,y)<\omega<\omega_R(x,y)\}$. Top left: $\omega=1$, with only an L cut-off. Top right: $\omega=2.25$, with the two cut-offs and the resonance. Bottom left : $\omega=3.35$, with a two-component UH resonance and a single component R cut-off. Bottom right : $\omega=4$, with a single component UH resonance and a two-component R cut-off.