# Variable coefficients and numerical methods for electromagnetic waves

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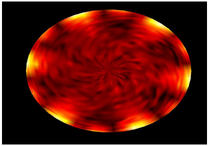


# Numerical methods for Helmholtz equation in inhomogeneous media

# The cold plasma mathematical model

## Maxwell's time-harmonic equations

$$curl\ curl\ \mathbf{E} - \left(\frac{\omega}{c}\right)^2 \varepsilon \mathbf{E} = 0$$

- $\varepsilon(x) = \begin{pmatrix} 1 - \frac{\omega_p^2(x)}{\omega^2 - \omega_c^2} & -i \frac{\omega_c \omega_p^2(x)}{\omega(\omega^2 - \omega_c^2)} & 0 \\ i \frac{\omega_c \omega_p^2(x)}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2(x)}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2(x)}{\omega^2} \end{pmatrix}$
- Background magnetic field along the $z$ axis
- $\omega_p^2(x) = \frac{e^2 n_e(x)}{\varepsilon_0 m_e}$ plasma frequency
- $n_e(x)$ plasma density
- $\omega_c = \frac{eB_0}{m_e}$ cyclotron frequency

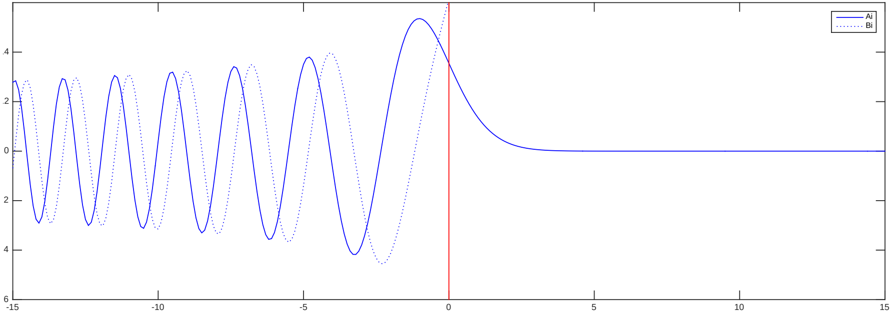
[Stix] **A general analysis of this model is able to provide a surprisingly comprehensive view of plasma waves.**

# Perpendicular propagation, parallel polarization
# O-mode equation and CutOff

## 2D Helmholtz equation for the total field

- $-\Delta u - \frac{\omega^2}{c^2}(1 - Cn_e(\mathbf{x}))u = 0$
- smooth variable coefficient : Cutoff $\Leftrightarrow 1 - Cn_e(\mathbf{x}) = 0$

## Airy function in 1D : $-u'' + xu = 0$




✓ $n_e < 1/C$ $\Rightarrow$ Propagating waves

▸ $n_e = 1/C$ $\Rightarrow$ Cutoff

✗ $n_e > 1/C$ $\Rightarrow$ Evanescent waves

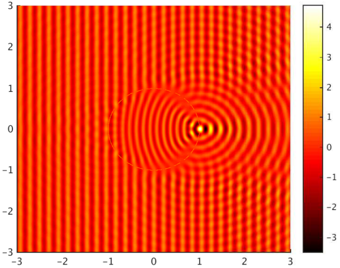
# Piecewise constant and inhomogeneous media

## Luneburg Lens

$$\text{Radius} \quad R = 1$$

$$\epsilon(\mathbf{x}) = 2 - |\mathbf{x}|^2/R^2$$

$$\mathcal{L} = -\Delta - \kappa^2\epsilon(\mathbf{x})$$




Piecewise constant approximation

Approximation using $\nabla\epsilon$

# Piecewise constant and inhomogeneous media

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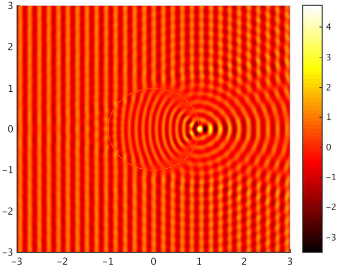
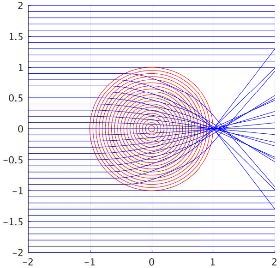
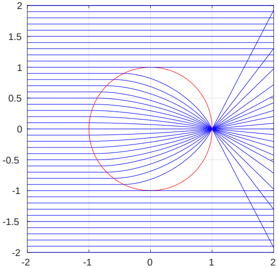
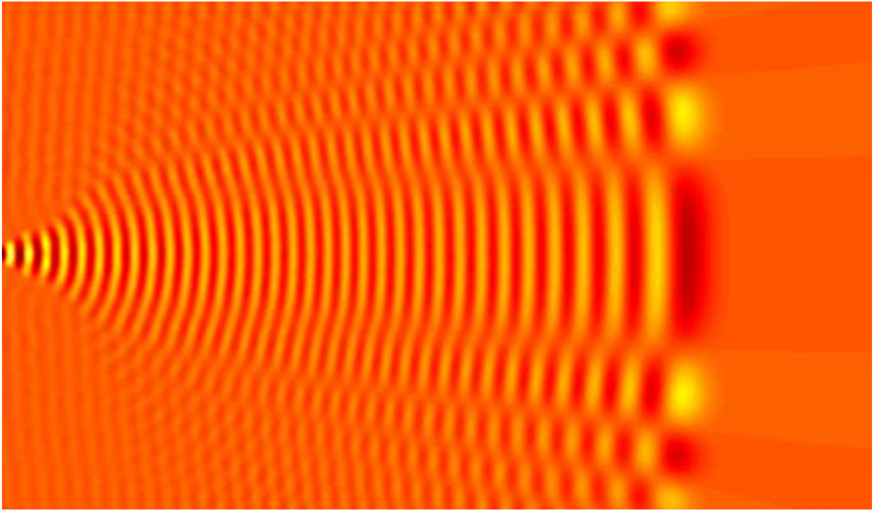



## Piecewise constant approximation




## Approximation using $\nabla\epsilon$





- Linear wave propagation in inhomogeneous media
- Continuous transition from propagating to evanescent medium
- Accurate numerical simulation of this transition

# The boundary value problem

## 2D Helmholtz equation for the total field

- $-\Delta u - \frac{\omega^2}{c^2}(1 - Cn_e(\mathbf{x}))u = 0$
- smooth variable coefficient
  $sign = \pm 1$, Cutoff $\Leftrightarrow 1 - Cn_e(\mathbf{x}) = 0$

## Artificially truncated domain

$\Omega \subset \mathbb{R}^2$ bounded domain

$$-\Delta u(\mathbf{x}) - \kappa^2 \epsilon(\mathbf{x}) u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega$$

Boundary conditions : metallic, absorbing, incoming

$$\begin{array}{ll} \partial_n u + \imath \kappa \vartheta u = g, & \mathbf{x} \in \partial\Omega^R \\ u = 0, & \mathbf{x} \in \partial\Omega^D \end{array}$$

# Plane Wave Methods
## For constant coefficients

Incorporate information about the solution
in the basis functions

- Discontinuous Enrichment method
  - C. Farhat, I. Harari, L.P. Franca, (2001).
- Partition of Unity Method
  - J.M. Melenk, I. Babuška, (1996).
- Trefftz type method
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**Novelty**

- Variable coefficients

**Challenges**

- Design basis functions
- Convergence of the method

# Trefftz Discontinuous Galerkin method

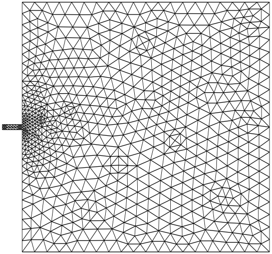
Helmholtz equation $\mathcal{L}u = 0$

$\mathcal{L}u := -\Delta u - \kappa^2 \epsilon(\mathbf{x}) u$

Mesh $\mathcal{T}_h = \{K\}$

**TdG space**

$\mathbb{H}_K = \{v \in L^2(K) : v$ smooth, $-\Delta v - \kappa^2 \epsilon(\mathbf{x}) v = 0\}$



Weak formulation If $\mathcal{L}u_K = 0$ in $\mathring{K}$

$$\int_K u_K \overline{(-\Delta_h v - \kappa^2 \epsilon v)} + \int_{\partial K} u_K \overline{\nabla_h v \cdot \mathbf{n}} - \int_{\partial K} \nabla_h u_K \cdot \mathbf{n} \overline{v} = 0 \quad \forall v \in \mathbb{H}_K$$

$$\text{Find } u \in \mathbb{H} := \prod_K \mathbb{H}_K \ s.t. \ \mathcal{A}_h(u, v) = \ell_h(v), \forall v \in \mathbb{H}$$

Discretization

Finite dimensional space $\mathbb{V}_K \subset \mathbb{H}_K$ ?

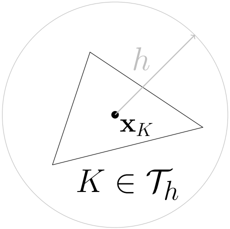
Remark $\dim(\mathbb{V}_K) = p$

# Generalized Plane Waves (GPWs) at a glance

- Smooth functions
- Associated with partial differential equation $\mathcal{L}u = 0$
- Introduced for variable coefficient operators
  - Local approximation
- *Generalization of classical PW*

## Goal

- High order approximation $u \approx u_a$




## Challenges to find $u_a$

- Design

$$\mathcal{L}u_a \approx 0$$

- Best approximation (Interpolation) properties

$$\forall\ u \text{ s.t. } \mathcal{L}u = 0, \exists\ u_a \text{ satisfying } \|u - u_a\| \leq Ch^n$$

# Local definition of a GPW

**Constant coefficient**

$$\mathcal{L} = -\Delta - \kappa^2 \epsilon_K$$

$$\varphi = \exp\left(\imath\kappa\sqrt{\epsilon_K}\mathbf{d}\cdot\mathbf{x}\right)$$

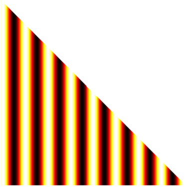
$$\Rightarrow \mathcal{L}\varphi = 0$$

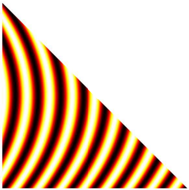
**Variable coefficient**

$$\mathcal{L} = -\Delta - \kappa^2 \epsilon(\mathbf{x})$$

$$\varphi = \exp\left(\imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}\mathbf{d}\cdot\mathbf{x} + \text{ H.O.T.}\right)$$

$$\Rightarrow \mathcal{L}\varphi \approx 0$$





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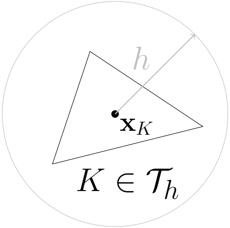
## Definition of a GPW

- For a given point $\mathbf{x}_K$
- For a partial differential operator $\mathcal{L}$
- For a parameter $q$




✓ $\varphi(\mathbf{x}) = \exp P(\mathbf{x})$

✓ $\begin{cases} \text{Taylor expansion at } \mathbf{x}_K \\ \mathcal{L}\varphi(\mathbf{x}) = O\left(h^q\right) \end{cases}$

# Building a GPW : The system

$$-\mathcal{L}\varphi(\mathbf{x}) = \left(\partial_x^2 P(\mathbf{x}) + (\partial_x P(\mathbf{x}))^2 + \partial_y^2 P(\mathbf{x}) + (\partial_y P(\mathbf{x}))^2 + \kappa^2 \epsilon(\mathbf{x})\right) e^{P(\mathbf{x})} = O\left(h^q\right)$$

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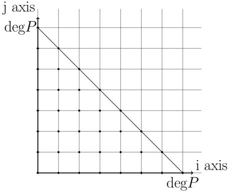
## The unknowns $\quad \mathbf{x} = (x, y)$




- $\varphi = \exp P$
- $P(\mathbf{x}) = \sum_{0 \leq i+j \leq \deg P} \lambda_{i,j} (x - x_K)^i (y - y_K)^j$

✓ $\deg P$ and $\{\lambda_{i,j}\}_{0 \leq i+j \leq \deg P}$ $\quad \Rightarrow N_{un} = \frac{(\deg P+1)(\deg P+2)}{2}$

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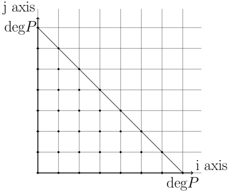


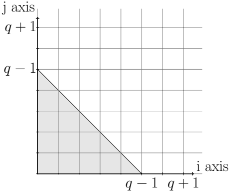

$\Rightarrow N_{un} = \frac{(\deg P+1)(\deg P+2)}{2}$

## The equations

- $\mathcal{L}\varphi(\mathbf{x}) = O\left(h^q\right)$

✓ $\forall (i,j)$ such that $0 \le i+j < q$

$$\partial_x^i \partial_y^j \left[\mathcal{L}(\exp P) / \exp P\right](\mathbf{x}_K) = 0$$




$\Rightarrow N_{eq} = \frac{q(q+1)}{2}$

# Building a GPW : The system

$$-\mathcal{L}\varphi(\mathbf{x}) = \left(\partial_x^2 P(\mathbf{x}) + (\partial_x P(\mathbf{x}))^2 + \partial_y^2 P(\mathbf{x}) + (\partial_y P(\mathbf{x}))^2 + \kappa^2 \epsilon(\mathbf{x})\right) e^{P(\mathbf{x})} = O\left(h^q\right)$$

## The unknowns

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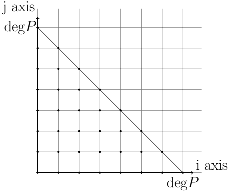


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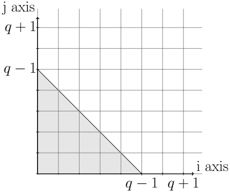



$$\Rightarrow N_{eq} = \frac{q(q+1)}{2}$$

Choice $\deg P = q + 1$

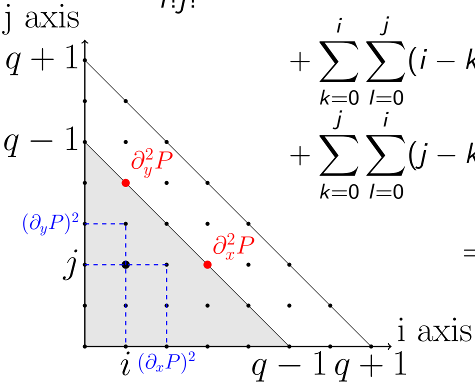
# Building a GPW : Structure of the non-linearity

$$\left[\partial_x^i\partial_y^j\left(\partial_x^2 P + (\partial_x P)^2 + \partial_y^2 P + (\partial_y P)^2\right)\right](\mathbf{x}_K)$$

## Identify linear and non-linear terms

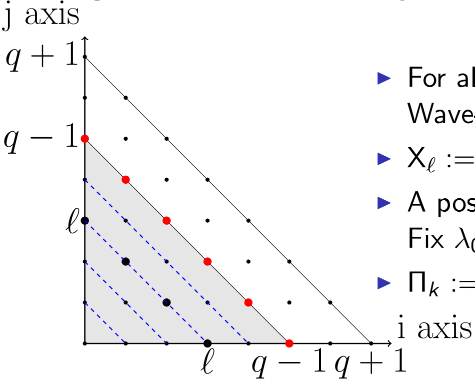
$\forall (i,j)\in\mathbb{N}^2$ such that $0\le i+j\le q-1$

$$-\kappa^2\frac{\partial_x^i\partial_y^j\epsilon(\mathbf{x}_K)}{i!j!} = (i+2)(i+1)\lambda_{i+2,j} + (j+2)(j+1)\lambda_{i,j+2}$$
$$+\sum_{k=0}^{i}\sum_{l=0}^{j}(i-k+1)(k+1)\lambda_{i-k+1,j-l}\lambda_{k+1,l}$$
$$+\sum_{k=0}^{j}\sum_{l=0}^{i}(j-k+1)(k+1)\lambda_{i-l,j-k+1}\lambda_{l,k+1}$$




$\Rightarrow$ Hierarchy of linear sub-systems of increasing size

# Building a GPW : Hierarchy of linear sub-systems




- For all $(i, j)$ such that $i + j = \ell$
  Wave-front structure
- $X_\ell := [\lambda_{0,\ell+2}, \dots, \lambda_{\ell+2,0}]^T$
- A possible option
  Fix $\lambda_{0,\ell+2}$ and $\lambda_{1,\ell+1}$
- $\Pi_k := (k+2)(k+1)$

$$\underbrace{\left[\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ \hline \Pi_0 & & \Pi_\ell & & \\ & \ddots & & \ddots & \\ & & \Pi_\ell & & \Pi_0 \end{array}\right]}_{L_\ell} \underbrace{\left[\begin{array}{c} \lambda_{0,\ell+2} \\ \lambda_{1,\ell+1} \\ \hline \lambda_{2,\ell} \\ \vdots \\ \lambda_{\ell+2,0} \end{array}\right]}_{X_\ell} = \underbrace{\left[\begin{array}{c} * \\ * \\ \hline \\ RHS \\ \\ \end{array}\right]}_{R_\ell}$$

$\Rightarrow$ Solve by **forward substitution**

$$L_\ell X_\ell = R_\ell$$

# Building a GPW : Algorithm

**Algorithm 1** Induction on the global degree $\ell = i + j$

Fix $\lambda_{0,0} = 0$, $(\lambda_{0,1}, \lambda_{1,0}) = \imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}\mathbf{d}$

**for** $\ell \leftarrow 0,\ q-1$ **do** $\qquad \triangleright\ q$

  Fix $\lambda_{0,\ell+2}$ and $\lambda_{1,\ell+1}$

  $\mathrm{R}_\ell \leftarrow f\left(\{\lambda_{i,j}\}_{i+j\leq\ell+1},\ \kappa, \left\{\partial_x^i\partial_y^j\epsilon(\mathbf{x}_K)\right\}_{i+j\leq\ell}\right) \qquad \triangleright\ \mathcal{L},\ \mathbf{x}_K$

  **for** $k \leftarrow 0,\ \ell$ **do**

    $\lambda_{k+2,\ell-k} := \dfrac{1}{\Pi_{\ell-k}}\left(\mathrm{R}_\ell[k+2] - \Pi_k\lambda_{k,\ell-k+2}\right) \qquad \triangleright\ \mathcal{L}$

$P(x,y) \leftarrow \sum_{0\leq i+j\leq q+1} \lambda_{i,j}(x-x_K)^i(y-y_K)^j \qquad \triangleright\ \mathbf{x}_K,\ q$

$\varphi(\mathbf{x}) \leftarrow \exp P(\mathbf{x})$

## Summary

- Analytic formula for $\lambda_{i,j}$ $\qquad \oslash\ h$
- $\mathcal{L}\varphi = [-\Delta - \kappa^2\epsilon(\mathbf{x})]\varphi = O(h^q)$ $\qquad \oslash\ \{\lambda_{i,j}\}_{i\in\{0,1\}}$

# Towards approximation properties

## Normalization : Choice of $\{\lambda_{i,j}\}_{i\in\{0,1\}}$

- $(\lambda_{0,1},\lambda_{1,0}) = \imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}\mathbf{d}$ with $\mathbf{d} = (\cos\theta, \sin\theta)$
- $\lambda_{i,j} = 0$ if $i+j \neq 1$ $\forall\ \theta$

## Local set of approximated solutions

$\forall \ell$ such that $1 \leq \ell \leq p$, $\theta_\ell = 2\pi\ell/p$

$$\Rightarrow \mathbb{V}_K = Span\{\varphi_\ell\}_{1\leq\ell\leq p}$$

$\triangleright$ $\mathbf{x}_K$, $p$, $q$

- $\varphi_\ell(x,y) = \exp\left(\imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}(\cos\theta_\ell(x-x_K)+\sin\theta_\ell(y-y_K)) + H.O.T\right)$
- $\Delta\varphi_\ell = \underbrace{\left(\partial_x^2 P_\ell + (\partial_x P_\ell)^2 + \partial_y^2 P_\ell + (\partial_y P_\ell)^2\right)}_{=-\kappa^2\epsilon+O(h^q)}\varphi$

# Best approximation properties

## Theorem

- $\forall n \in \mathbb{N},\ n > 0$
- $u \in \mathcal{C}^{n+1}$ such that $\mathcal{L}u = 0$, and $\epsilon \in \mathcal{C}^n$
- $p = 2n + 1$ basis functions
- $q = n + 1$
- $\epsilon(\mathbf{x}_K) \neq 0$
- Build $\mathbb{V}_K$ so that $\mathcal{L}\varphi = O(h^q)\ \forall \varphi \in \mathbb{V}_K$
- $\exists\ u_a \in \mathbb{V}_K$ such that

$$\left\{ \begin{array}{l} \left|u\left(\mathbf{x}\right) - u_a\left(\mathbf{x}\right)\right| \leq C(n) \left|\mathbf{x} - \mathbf{x}_K\right|^{n+1} \left\|u\right\|_{\mathcal{C}^{n+1}} \\ \left|\nabla u\left(\mathbf{x}\right) - \nabla u_a\left(\mathbf{x}\right)\right| \leq C(n) \left|\mathbf{x} - \mathbf{x}_K\right|^{n} \left\|u\right\|_{\mathcal{C}^{n+1}} \end{array} \right.$$

## Numerical results

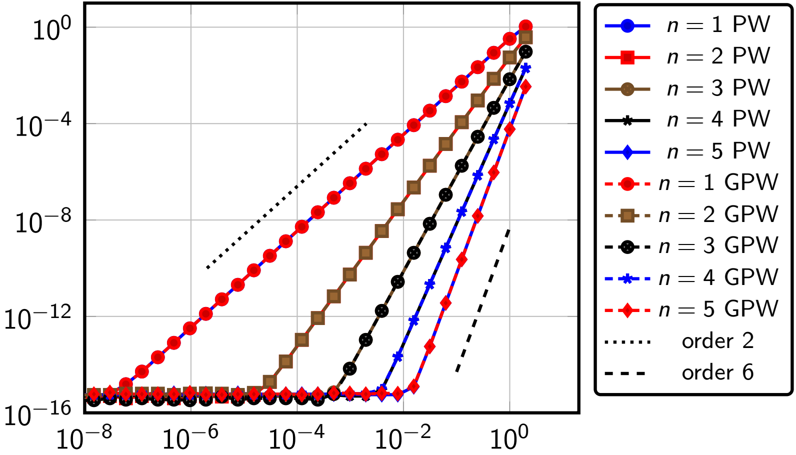
$$\left\|u - u_a\right\|_{L^\infty\left(\{\mathbf{x}\in\mathbb{R}^2, |\mathbf{x}-\mathbf{x}_K|<h\}\right)} = O\left(h^{n+1}\right)$$

# PW and GPW for constant coefficient $\mathcal{L} = -\Delta - 1$ and $u = e^{\imath y}$

$\mathbf{x}_K = (-3, -1)$

with $p = 2n + 1$ basis functions
at the order $q = n + 1$

# PW and GPW for constant coefficient $\mathcal{L} = -\Delta - 1$ and $u = e^{\imath y}$

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# PW and GPW for variable coefficient $\mathcal{L} = -\Delta + (x-1)$ and $u = Ai(x)e^{\imath y}$

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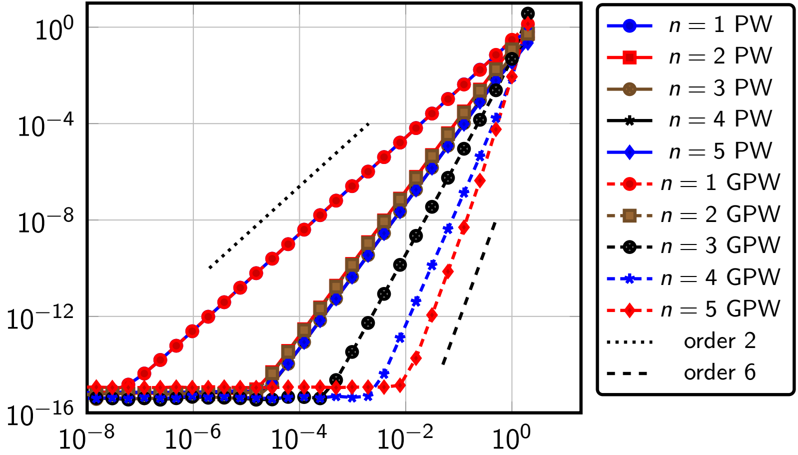
# PW and GPW for variable coefficient $\mathcal{L} = -\Delta + (x-1)$ and $u = Ai(x)e^{\imath y}$




$\mathbf{x}_K = (-3, -1)$

with $p = 2n + 1$ basis functions
at the order $q = n + 1$

# Convergence of the GPW+TdG method

## GPW space $\mathbb{V}_K$ [IG & Després '14]

$$\mathbb{V}_K \not\subset \mathbb{H}_K = \{v \in L^2(K) : v \text{ smooth} \\ \text{and } -\Delta_h v - \kappa^2 \epsilon v = 0, \text{ on } K\}$$

## Stabilized formulation

$$\mathcal{B}_h(u,v) = \mathcal{A}_h(u,v) + \imath\kappa \int_\Omega \gamma(\Delta_h u + \kappa^2 u)\overline{(\Delta_h v + \kappa^2 v)} dS$$

## Coercivity and continuity

$$\|u\|_{DG}^2 \leq |\mathcal{B}_h(u,u))|$$

$$|\mathcal{B}_h(u,v))| \leq C\|u\|_{DG+}\|v\|_{DG}$$

## Theorem (smooth solution) [IG & Monk '16]

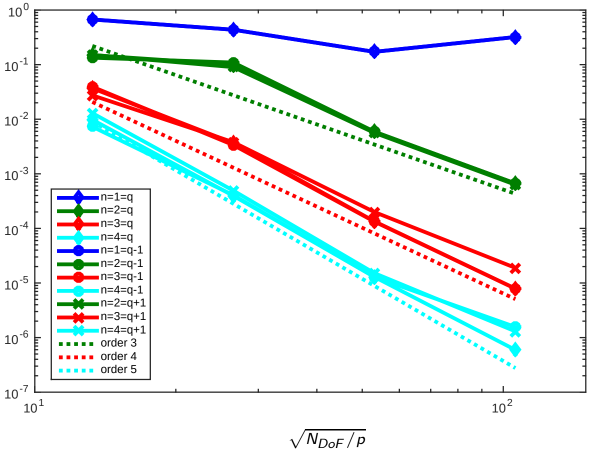
$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^n$$

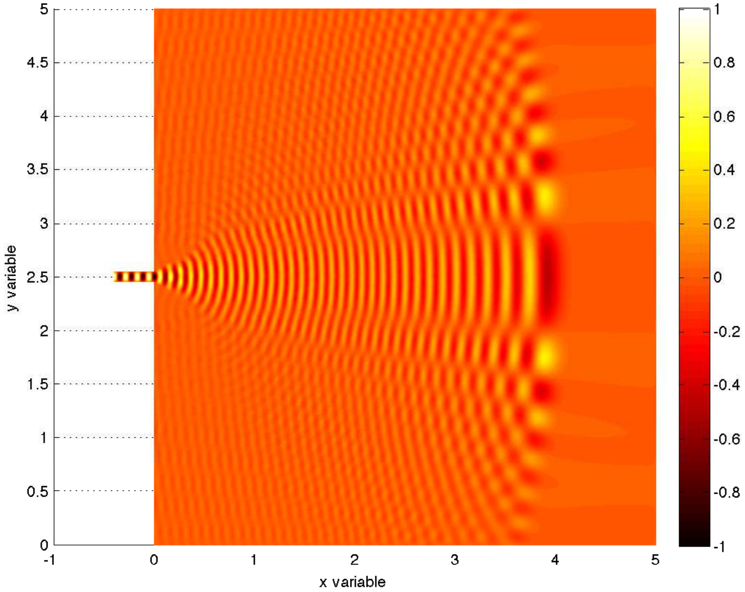
provided that $p = 2n+1$, $n \geq 2$, $q = n+1$ and for given flux parameters

# GPW + TdG

2D test case : $L^2(\Omega)$ norm convergence

$u_{ex}(\mathbf{x}) = Ai(\kappa^{2/3}y)$, $\kappa = 15$




y variable
x variable
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-1
0
1
2
3
4
5
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

# WKB expansion for Helmholtz equation

$$-\Delta u - \kappa^2 \epsilon(\mathbf{x}) u = 0$$

Ansatz : $\varphi(\mathbf{x}) = A(\mathbf{x}) \exp \imath\kappa S(\mathbf{x})$

$$\begin{aligned}(-\Delta - \kappa^2\epsilon(\mathbf{x}))\varphi(\mathbf{x}) \quad &= \kappa^2 A(\mathbf{x})\Big(|\nabla S(\mathbf{x})|^2 - \epsilon(\mathbf{x})\Big) e^{\imath\kappa S(x)} \\ &\quad - \imath\kappa\Big(A(\mathbf{x})\Delta S(\mathbf{x}) + \nabla A(\mathbf{x}) \cdot \nabla S(\mathbf{x})\Big) e^{\imath\kappa S(\mathbf{x})} \\ &\quad - \Big(\Delta A(\mathbf{x})\Big) e^{\imath\kappa S(\mathbf{x})}\end{aligned}$$

- $O(\kappa^2)$ terms - Eikonal equation for $S$
  $$\Rightarrow |\nabla S(\mathbf{x})|^2 - \epsilon(\mathbf{x}) = 0$$
- $O(\kappa^1)$ terms - Transport equation
  $$\Rightarrow \nabla \cdot \big(A(\mathbf{x}) \nabla S(\mathbf{x})\big) = 0$$
- Neglect higher order terms

$\Rightarrow A, S$ are independent of $\kappa$

**Towards variable amplitude GPWs**

# Back to GPWs

$$\mathcal{L} = -\Delta - \kappa^2 \epsilon(\mathbf{x})$$

Ansatz : $\varphi(x) = A(\mathbf{x}) \exp \imath\kappa S(\mathbf{x})$

$$\begin{aligned}
(-\Delta - \kappa^2\epsilon(\mathbf{x}))\varphi(\mathbf{x}) \quad &= \kappa^2 A(\mathbf{x})\Big(|\nabla S(x)|^2 - \epsilon(\mathbf{x})\Big)e^{\imath\kappa S(\mathbf{x})} \\
&- \imath\kappa\Big(A(\mathbf{x})\Delta S(\mathbf{x}) + \nabla A(\mathbf{x}) \cdot \nabla S(\mathbf{x})\Big)e^{\imath\kappa S(\mathbf{x})} \\
&- \Big(\Delta A(\mathbf{x})\Big)e^{\imath\kappa S(\mathbf{x})}
\end{aligned}$$

# Back to GPWs

$$\mathcal{L} = -\Delta - \kappa^2 \epsilon(\mathbf{x})$$

Ansatz : $\varphi(x) = A(\mathbf{x}) \exp \imath\kappa S(\mathbf{x})$

$$\begin{aligned}(-\Delta - \kappa^2\epsilon(\mathbf{x}))\varphi(\mathbf{x}) \quad &= \kappa^2 A(\mathbf{x})\Big(|\nabla S(x)|^2 - \epsilon(\mathbf{x})\Big)e^{\imath\kappa S(\mathbf{x})} \\ &- \imath\kappa\Big(A(\mathbf{x})\Delta S(\mathbf{x}) + \nabla A(\mathbf{x})\cdot\nabla S(\mathbf{x})\Big)e^{\imath\kappa S(\mathbf{x})} \\ &- \Big(\Delta A(\mathbf{x})\Big)e^{\imath\kappa S(\mathbf{x})}\end{aligned}$$

To get an approximate solution

- $O(\kappa^2)$ terms - Eikonal equation for $S$

$$\Rightarrow |\nabla S(\mathbf{x})|^2 - \epsilon\left(\mathbf{x}_G\right) = 0$$

- Gather the remaining terms - Equation for $A$

$$\Delta A(x) + \imath\kappa\nabla\cdot\left(A(\mathbf{x})\nabla S(\mathbf{x})\right) + \kappa^2 A(\mathbf{x})\Big(|\nabla S(\mathbf{x})|^2 - \epsilon(\mathbf{x})\Big) = O(h^q)$$

$$\Rightarrow \mathcal{L}\varphi \approx 0 \text{ independently of } \kappa$$

# WKB for a vector equation

## WKB ansatz

$$\overrightarrow{\psi}(x) = \mathbf{A}(\mathbf{x})e^{\imath\kappa S(\mathbf{x})}$$

## $O(\kappa^2)$ terms

$$\mathcal{M}(S(\mathbf{x}))\mathbf{A}(\mathbf{x})$$

For a matrix $\mathcal{M}$
- independent of $\mathbf{A}$
- depending on the coefficients of the PDE

## WKB

- Determinant of $\mathcal{M}$
  $\Rightarrow$ Solved for $S$, independent of $\mathbf{A}$
- Eigenvectors of $\mathcal{M}$
  $\Rightarrow$ Solved for $\mathbf{A}$ for a given $S$
- Neglect higher order terms

$\Rightarrow$ $\mathbf{A}$, $S$ are independent of $\kappa$

# Back to GPWs for Maxwell's equation

$$\mathcal{L}\mathbf{v} \equiv \nabla \times \nabla \mathbf{v} - \kappa^2 \epsilon \mathbf{v}$$

Ansatz : $\overrightarrow{\psi}(x) = \mathbf{A}(\mathbf{x}) \exp \imath\kappa S(\mathbf{x})$

$$\begin{aligned} \mathcal{L}\overrightarrow{\psi} &= \kappa^2 \left[-\nabla S \times (\nabla S \times \mathbf{A}) - \epsilon \mathbf{A}\right] e^{\imath\kappa S} \\ &\quad + \imath\kappa \left[\nabla S \times (\nabla \times \mathbf{A}) + \nabla \times (\nabla S \times \mathbf{A})\right] e^{\imath\kappa S} \\ &\quad + \left[\nabla \times \nabla \times \mathbf{A}\right] e^{\imath\kappa S} \end{aligned}$$

# Back to GPWs for Maxwell's equation

$$\mathcal{L}\mathbf{v} \equiv \nabla \times \nabla \mathbf{v} - \kappa^2 \epsilon \mathbf{v}$$

Ansatz : $\overrightarrow{\psi}(x) = \mathbf{A}(\mathbf{x}) \exp \imath\kappa S(\mathbf{x})$

$$\begin{aligned} \mathcal{L}\overrightarrow{\psi} \quad &= \kappa^2 \left[-\nabla S \times (\nabla S \times \mathbf{A}) - \epsilon \mathbf{A}\right] e^{\imath\kappa S} \\ &+ \imath\kappa \left[\nabla S \times (\nabla \times \mathbf{A}) + \nabla \times (\nabla S \times \mathbf{A})\right] e^{\imath\kappa S} \\ &+ \left[\nabla \times \nabla \times \mathbf{A}\right] e^{\imath\kappa S} \end{aligned}$$

To get an approximate solution $\Rightarrow \mathcal{L}\overrightarrow{\psi} \approx 0$ independently of $\kappa$

- $O(\kappa^2)$ terms - Eikonal equation for $S$
$$\Rightarrow \det\left(\nabla S \cdot \nabla S^T - |\nabla S|^2 Id - \epsilon\right) = 0 \text{ at } \mathbf{x}_G$$
- Gather the remaining terms - Equations for $A$

$$\nabla \times \nabla \times \mathbf{A} + \imath\kappa \nabla \times (\nabla S \times \mathbf{A})$$

$$+ \imath\kappa \nabla S \times (\nabla \times \mathbf{A}) - \kappa^2 \nabla S \times (\nabla S \times \mathbf{A}) - \kappa^2 \epsilon \mathbf{A} = O(h^q)$$

# Design of a WKB-GPW : The system

$$\mathcal{L}\overrightarrow{\psi}(\mathbf{x}) = \left(\nabla \times \nabla \times -\kappa^2\epsilon(\mathbf{x})\right)\left[\mathbf{A}(\mathbf{x})\exp \imath\kappa S(\mathbf{x})\right] = O(h^q)$$

## The unknowns

- $\overrightarrow{\psi} = \mathbf{A}\exp \imath\kappa S$
- $A_\alpha(\mathbf{x}) = \sum_{0\leq i+j+k\leq D} \lambda^\alpha_{i,j,k}(x-x_G)^i(y-y_G)^j(z-z_G)^k$

✓ $D$ and $(\lambda^x_{i,j,k}, \lambda^y_{i,j,k}, \lambda^z_{i,j,k})_{\{0\leq i+j+k\leq D\}}$ $\Rightarrow N_{un} = 3\frac{(D+1)^2 D}{2}$

## The equations

- $\mathcal{L}\overrightarrow{\psi}(\mathbf{x}) = O\left(h^q\right)$

✓ $\forall(i,j,k)$ such that $0 \leq i+j+k \leq q-1$

$$\partial_x^i\partial_y^j\partial_z^k\left[\mathcal{L}(\mathbf{A}\exp \imath\kappa S)/\exp \imath\kappa S\right](\mathbf{x}_G) = 0$$

$$\Rightarrow N_{eq} = 3\frac{q^2(q-1)}{2}$$

Choice $D = q+1$

# Design of a GPW : Structure of the system

$$\Big[\partial_x^i\partial_y^j\partial_z^k\Big(\nabla\times\nabla\times\mathbf{A}+\imath\kappa\nabla\times(\nabla S\times\mathbf{A})$$

$$+\imath\kappa\nabla S\times(\nabla\times\mathbf{A})-\kappa^2\nabla S\times(\nabla S\times\mathbf{A})-\kappa^2\epsilon\mathbf{A}\Big)\Big](\mathbf{x}_G)$$

## Hierarchy of linear sub-systems of increasing size

- Equations for $i+j+k=\ell$

$$\partial_x^i\partial_y^j\partial_z^k\Big(\nabla\times\nabla\times\mathbf{A}\Big)(\mathbf{x}_G)=RHS_{i,j,k}$$

- Unknowns $(\lambda^x_{r,s,t},\lambda^y_{r,s,t},\lambda^z_{r,s,t})_{\{r+s+t=\ell+2\}}$
- Blocs of 3 coupled equations

$$\nabla\times\nabla\times\mathbf{A}=-\Delta\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})$$

# Summary

## WKB inspired GPWs

For a scalar or vector operator $\mathcal{L}$

## Ansatz

$$\varphi(x) = A(x) \exp \imath\kappa S(x)$$
$$\overrightarrow{\psi}(x) = \mathbf{A}(x) \exp \imath\kappa S(x)$$

## From lowest order terms

- Equation for $S$
- ✓ Phase function $S$ independent of $\kappa$

## Remaining terms

- Approximate equation for $A/\mathbf{A}$
- ✗ Amplitude function $A/\mathbf{A}$ not independent of $\kappa$

# Future work

## GPWs

- High frequency regime $h \propto \lambda$
- Extension to 3D scalar case
  - Collaboration with G. Sylvand (Airbus)
- Focus on vector valued equations

## Trefftz DG + GPWs

- $h$-convergence and $p$-convergence
  - Collaboration with R. Hiptmair (ETH Zurich)
- Parallel implementation for the 3D Helmholtz equation
  - Collaboration with G. Stadler (NYU)
- Towards Parallel implementation for the cold plasma model

# Pseudo-spectral methods for the PDEs on surfaces of genus one

# Partial differential methods on surfaces

## Scalar valued partial differential equations

- Finite Elements
  - G. Dziuk and C. Elliott (2013).
- Finite Volumes
  - D. Calhoun, C. Helzel, R. Leveque (2008).
- Implicit representation of the surface
  - Level set approach
    D. Adalsteinsson, J. Sethian (1997).
  - Closest point method
    C. Macdonald, J. Brandman, S. Ruuth (2011).

# Partial differential methods on surfaces

## Vector valued partial differential equations

- ➢ Spherical harmonic expansions [Restricted to the sphere]
  - Both in the Physics and the Math literature
- ➢ Diffuse interface approach
  - A. Ratz, A. Voigt (2006).
- ➢ Discrete Exterior Calculus
  - A. Hirani (2003).
- ☞ Spectral methods

# Partial differential methods on surfaces

## Vector valued partial differential equations

- Spherical harmonic expansions [Restricted to the sphere]
  - Both in the Physics and the Math literature
- Diffuse interface approach
  - A. Ratz, A. Voigt (2006).
- Discrete Exterior Calculus
  - A. Hirani (2003).
- Spectral methods

### Novelty

- Genus 1 surfaces

### Challenges

- Discretization
- Harmonic vector fields

# Differential operators on surfaces $\quad \mathbf{x} \in \mathbb{R}^3$

Notation $G = \begin{pmatrix} G_{uu} & G_{uv} \\ G_{uv} & G_{vv} \end{pmatrix}$ with $G_{ij} = \partial_i \mathbf{x} \cdot \partial_j \mathbf{x}$ and $\det G = g$

For tangential vector fields $\mathbf{F} = F^1 \partial_u \mathbf{x} + F^2 \partial_v \mathbf{x}$, scalar fields $f$

$$Grad\, f = \nabla_\Gamma f = \left[ \frac{G_{vv}}{g} \partial_u f - \frac{G_{uv}}{g} \partial_v f \right] \partial_u \mathbf{x} + \left[ -\frac{G_{uv}}{g} \partial_u f + \frac{G_{uu}}{g} \partial_v f \right] \partial_v \mathbf{x}$$

$$Div(\mathbf{F}) = \nabla_\Gamma \cdot \mathbf{F} = \frac{1}{\sqrt{g}} \left[ \partial_u \left( \sqrt{g} F^1 \right) + \partial_v \left( \sqrt{g} F^2 \right) \right]$$

$$\Delta_\Gamma f = \frac{1}{\sqrt{g}} \left[ \partial_u \left( \frac{G_{vv}}{\sqrt{g}} \partial_u f - \frac{G_{uv}}{\sqrt{g}} \partial_v f \right) + \partial_v \left( \frac{G_{uu}}{\sqrt{g}} \partial_v f - \frac{G_{uv}}{\sqrt{g}} \partial_u f \right) \right]$$

$$\begin{aligned} Curl\, \mathbf{F} &= -\nabla_\Gamma \cdot (\mathbf{n} \times \mathbf{F}) \\ &= \frac{1}{\sqrt{g}} \left( \partial_u \left( G_{uv} F^1 + G_{vv} F^2 \right) - \partial_v \left( G_{uu} F^1 + G_{uv} F^2 \right) \right) \end{aligned}$$

# Pseudo-spectral discretization

## Fourier coefficients - Genus 1 surfaces

$u \in [0, 2\pi]$ and $v \in [0, 2\pi]$

### Fourier coefficients

$$[\mathcal{F}(f)]_{mn} \equiv \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(u,v) e^{-\imath(mu+nv)} du\, dv$$

### Function discretization

$$\vec{f} = \{f(u_k, v_l)\} \text{ on an uniform grid with } N^2 \text{ points}$$

$$\widehat{f}_{mn} = \left(\frac{h}{2\pi}\right)^2 \sum_{k=1}^{N} \sum_{l=1}^{N} f(u_k, v_l) e^{-\imath(mu_k+nv_l)}$$

### Operators discretization

- Derivation $\Rightarrow$ In Fourier space
- Multiplication $\Rightarrow$ In physical space

Notation : $Grad_h$, $Div_h$, $Curl_h$, $\Delta_{\Gamma,h}$

# Hodge decomposition

## The orthogonal decomposition

$$\mathbf{j} = \nabla_\Gamma \alpha + \mathbf{n} \times \nabla_\Gamma \beta + \mathbf{j}_H, \text{ with } \alpha \ , \ \beta \text{ scalar functions}$$

$\nabla_\Gamma \alpha$ curl free component
$\mathbf{n} \times \nabla_\Gamma \beta$ divergence free component
$\mathbf{j}_H$ harmonic vector field

## Computing the decomposition

- Solve $\Delta_\Gamma \alpha = \nabla_\Gamma \cdot \mathbf{j}$
- Solve $\Delta_\Gamma \beta = -\nabla_\Gamma \cdot (\mathbf{n} \times \mathbf{j})$
- Compute a basis of harmonic vector fields $\{\mathbf{h}_1, \mathbf{h}_2\}$
- Solve $\begin{pmatrix} \langle \mathbf{h}_1, \mathbf{h}_1 \rangle_\Gamma & \langle \mathbf{h}_1, \mathbf{h}_2 \rangle_\Gamma \\ \langle \mathbf{h}_2, \mathbf{h}_1 \rangle_\Gamma & \langle \mathbf{h}_2, \mathbf{h}_2 \rangle_\Gamma \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle \mathbf{h}_1, \mathbf{j} \rangle_\Gamma \\ \langle \mathbf{h}_2, \mathbf{j} \rangle_\Gamma \end{pmatrix}$

➢ $\mathbf{j} = \nabla_\Gamma \alpha + \mathbf{n} \times \nabla_\Gamma \beta + c_1 \mathbf{h}_1 + c_2 \mathbf{h}_2$

# Harmonic vector fields on a surface of genus 1

## Definition of the space of dimension 2

$$\begin{aligned} Div(\mathbf{F}) = \nabla_\Gamma \cdot \mathbf{F} = 0 \\ Curl(\mathbf{F}) = \nabla_\Gamma \cdot (\mathbf{n} \times \mathbf{F}) = 0 \end{aligned}$$

Remark : $\mathbf{F}$ and $\mathbf{n} \times \mathbf{F}$

## At the discrete level

Find the nullspace of the $2N^2 \times 2N^2$ matrix

$$\begin{cases} Div_h(\mathbf{F}) = 0 \\ Curl_h(\mathbf{F}) = 0 \end{cases}$$

$\Rightarrow$ Basis of harmonic vector fields $\{\mathbf{h}_1, \mathbf{h}_2\}$

## Randomized method for rank-deficient linear systems

[J. Sifuentes, Z. Gimbutas, L. Greengard (2015)]

$$(A_h + \vec{r}\vec{s}^T)\vec{y} = A_h\vec{q} \Rightarrow A_h(\vec{y} - \vec{q}) = 0$$

# The Laplace-Beltrami equation

$$\Delta_\Gamma \phi = b \quad (*)$$

## At the continuous level

$$\mathcal{M}_\Gamma = \{f : \Gamma \to \mathbb{R} \,|\, \langle f, e \rangle_\Gamma = 0\} \text{ with } e(u,v) = 1$$

$(*)$ invertible from $\mathcal{M}_\Gamma$ to $\mathcal{M}_\Gamma$
$\Rightarrow \forall b \in \mathcal{M}_\Gamma$, $\exists! \phi \in \mathcal{M}_\Gamma$ such that

$$\Delta_\Gamma \phi + \langle e, \phi \rangle_\Gamma \, e = b$$

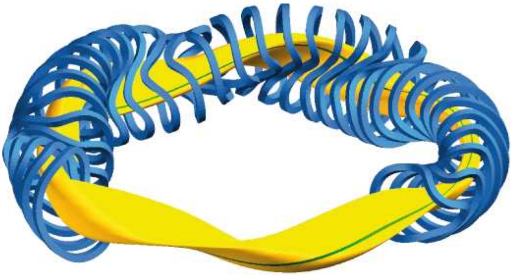
## At the discrete level

Solve

$$\left(\Delta_{\Gamma,h} + \vec{c}\,\vec{c}^{\,T}\right) \vec{\phi} = \vec{b}$$

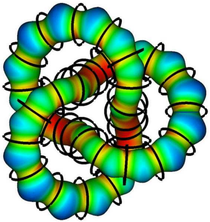
Preconditioner : $\left(\Delta_{I,h} + \vec{1}\,\vec{1}^{\,T}\right)^{-1}$

# Application to plasma physics



Stellarator

Knotatron
Courtesy of S. Hudson



# 3D parametrization of a Stellarator geometry

## Garabedian coordinates

$\mathbf{x}(s,u,v) =$

$$\begin{cases} \cos v \Big( r_0(v) + R(s,u,v) \left( \sum \Delta_{m,n} \cos((1-m)u + nv) - r_0(v) \right) \Big), \\ \sin v \Big( r_0(v) + R(s,u,v) \left( \sum \Delta_{m,n} \cos((1-m)u + nv) - r_0(v) \right) \Big), \\ z_0(v) + R(s,u,v) \left( \sum \Delta_{m,n} \sin((1-m)u + nv) - z_0(v) \right). \end{cases}$$

- Poloidal and toroidal angles $u \in [0, 2\pi]$, $v \in [0, 2\pi]$
  Radius-like parameter $s \in [0, 1]$
- Magnetic axis $(r_0(v), z_0(v))$
- Outer surface
  $(\sum \Delta_{m,n} \cos((1-m)u + nv), \sum \Delta_{m,n} \sin((1-m)u + nv))$
- Stretching function $R(s,u,v)$
- Nested genus 1 surfaces for increasing values of $s$

# 3D parametrization of a Knotatron geometry

## $(p, q)$-knot coordinates

Parametrized knot curve defined by $\mathbf{X}_L(v)$ as

$$\begin{cases} x_L(v) = \cos(pv)(R_0 + r_1\cos(qv)), \\ y_L(v) = \sin(pv)(R_0 + r_1\cos(qv)), \\ z_L(v) = r_1\sin(qv). \end{cases}$$

- Tangent unit tangent to the curve $\mathbf{T}(v)$
- Normal unit vector $\mathbf{N}(v) = (\cos(pv)\cos(qv), \sin(pv)\cos(qv), \sin(qv))$
- Binormal unit vector $\mathbf{B}(v) = \mathbf{T}(v) \times \mathbf{N}(v)$

The desired surface is then defined as

$$\mathbf{X}(s, u, v) = \mathbf{X}_L(v) + \frac{s}{a}\cos u\,\mathbf{N}(v) + \frac{s}{b}\sin u\,\mathbf{B}(v).$$

# The code

### Input

- Geometry $\mathbf{x}$
- Component of the metric tensor $G_{ij}$
- and their first and second order derivatives

### Operators

- From point values to point values
- FFT based

### Numerical solve

- $O(N^2 \log N)$ for $N^2$ points to represent the surface
- Iterative methods Bicg stab

### Implementation

- Fortran

Remarks

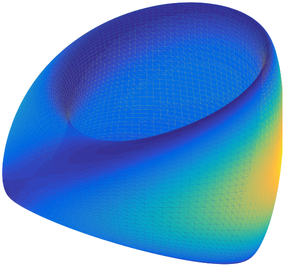
- Standard geometry input
  Independent from the rest of the code
- Global surface parametrization

# A Stellarator geometry

## Parameter values

- $r_0(v) = 4.8 + 0.1 \cos v$
- $z_0(v) = 0.1 \sin v$
- $s = 0.8$
- $R(s, u, v) = s(1 + 0.01(1 - s) \cos u \sin v)$

| m\n | -1 | 0 | 1 |
|---|---|---|---|
| -1 | 0.17 | 0.11 | 0 |
| 0 | 0 | 1 | 0.07 |
| 1 | 0 | 4.5 | 0 |
| 2 | 0 | -0.25 | - 0.45 |



# The Laplace-Beltrami equation on our Stellarator geometry

## Parameter values

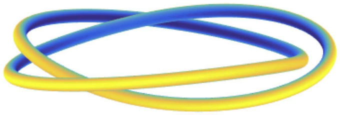
- $\psi(u, v) = e^{\cos u + \sin v} + e^{\cos(\kappa(u-v))}$
- $\psi_0 = \psi - \frac{\langle \psi, e \rangle_\Gamma}{\langle e, e \rangle_\Gamma} e$
- $b = \Delta_\Gamma \psi_0$
- $\kappa = 12$

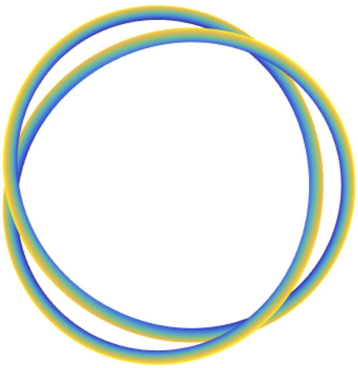
## Numerical results

Error $\|\phi_h - \psi_0\|_{2,\Gamma}$

| N | Iterations | Time | Error |
|---|---|---|---|
| 47 | 99 | 0.23E+01 | 0.12E+02 |
| 95 | 125 | 0.97E+01 | 0.36E-02 |
| 191 | 136 | 0.91E+02 | 0.11E-06 |
| 383 | 121 | 0.54E+03 | 0.42E-12 |
| 767 | 114 | 0.86E+03 | 0.44E-12 |

# A $(2, 3)$-knot geometry





Parameters :

- $R_0 = 10$ & $r_1 = 1$
- $p = 2$ & $q = 3$
- $a = b = 2$
- $s = 0.9$

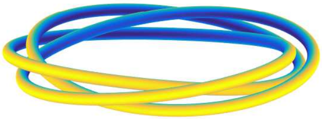
# Harmonic vector fields on our $(2,3)$-knot surface

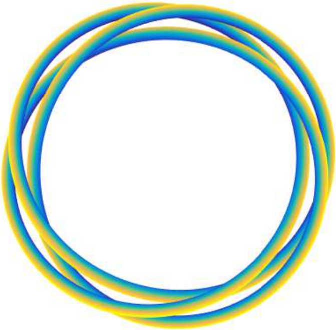
## Numerical results

Error $\|Div_h\vec{\psi}\|_{2,\Gamma}$ and $\|Curl_h\vec{\psi}\|_{2,\Gamma}$

| N | Iterations | Time | Div error | Curl error |
|---|---|---|---|---|
| 47 | 676 | 0.23E+02 | 0.49E-09 | 0.49E-09 |
| 95 | 532 | 0.53E+02 | 0.63E-10 | 0.17E-09 |
| 191 | 607 | 0.70E+03 | 0.56E-10 | 0.11E-09 |
| 383 | 658 | 0.50E+04 | 0.10E-08 | 0.13E-08 |
| 767 | 686 | 0.69E+04 | 0.12E-09 | 0.38E-09 |

# A $(3, 4)$-knot geometry





Parameters :

- $R_0 = 10$ & $r_1 = 1$
- $p = 3$ & $q = 4$
- $a = b = 2$
- $s = 0.9$

# Hodge decomposition on our $(3,4)$-knot surface

## Parameter values

- $\mathbf{j}(\mathbf{x}) = \nabla\left(\sin \imath\kappa\mathbf{k}\cdot\mathbf{x}\right) + \mathbf{n}\times\nabla\left(\sin \imath\kappa\mathbf{k}\cdot\mathbf{x}\right)$
- $\kappa = 10$
- $\mathbf{k} = (0,0,1)$
- $\mathbf{j}_T = \mathbf{j} - (\mathbf{j}\cdot\mathbf{n})\mathbf{n}$

## Numerical results

Error $\|\mathbf{j}_T - \nabla_\Gamma\alpha - \mathbf{n}\times\nabla_\Gamma\beta - \mathbf{j}_H\|_2$

| N | Iterations | Time | Error |
|---|---|---|---|
| 47 | 867 | 0.14E+03 | 0.49E-02 |
| 95 | 839 | 0.70E+03 | 0.94E-02 |
| 191 | 967 | 0.56E+04 | 0.10E-04 |
| 383 | 1056 | 0.37E+05 | 0.11E-06 |
| 767 | 1181 | 0.93E+05 | 0.11E-08 |

# Future work

## Electromagnetic scattering : Gen. Debye sources [Epstein et al. '10,'13]

$$\mathbf{E} = \imath k \mathbf{A} - \nabla \phi - \nabla \times \mathbf{A}_m$$

$$\mathbf{B} = \nabla \times \mathbf{A} + \imath k \mathbf{A}_m - \nabla \phi_m$$

- Potentials and Antipotentials Symmetry, Decouples at zero frequency
- Combined Source Integral Equation method

$$\mathbf{A}_{(m)}(\mathbf{x}) = \int_\Gamma g_k(\mathbf{x}-\mathbf{y})\mathbf{j}_{(m)}(\mathbf{y}) dA_y \;\&\; \phi_{(m)}(\mathbf{x}) = \int_\Gamma g_k(\mathbf{x}-\mathbf{y}) q_{(m)}(\mathbf{y}) dA_y$$

- Laplace-Beltrami equations for surface unknowns

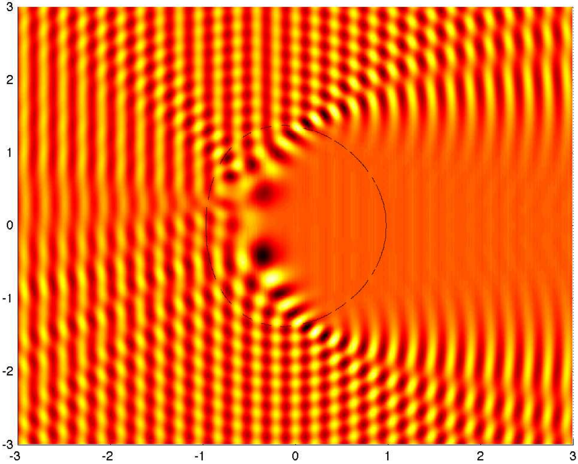
## Taylor states or Beltrami fields [O'neil Cerfon '16]

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}$$

## Anisotropic Maxwell equations

- Volume integral representation

# Thank you for your attention

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